Image Analysis and Processing

Image Enhancements in the Frequency Domain

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Book

Chapter 4 (pages 147–219)

- Digital Image Processing, Second Edition
  - authors: Rafael C. Gonzalez and Richard E. Woods
  - editor: Prentice Hall

Frequency Domain Filtering Operation

- Frequency domain: space defined by values of the Fourier transform and its frequency variables (u, v).
- Relation between Fourier Domain and image:
  - $u = v = 0$ corresponds to the gray-level average
  - Low frequencies: image's component with smooth gray-level variation (*e.g.* areas with low variance)

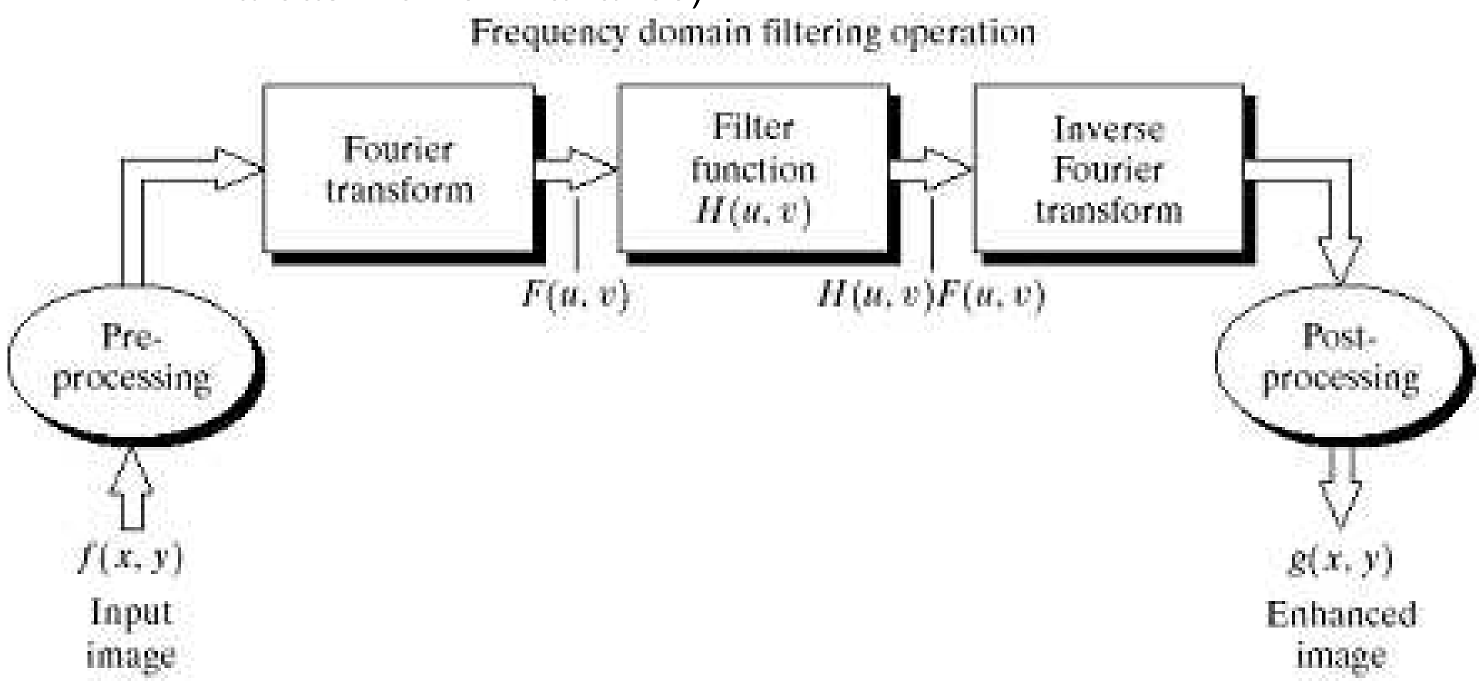


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Filtering in the Frequency Domain

Let $H(u,v)$ a *filter*, also called *filter transfer function*.

- Filter: suppress certain frequencies while leaving others unchanged

$$G(u,v) = H(u,v)\, F(u,v)$$

- $H(u,v)$ in image processing:
  - In general $H(u,v)$ is real: zero-phase-shift filter
  - H multiply real and imaginary parts of F
  - $\Phi(u,v) = \tan^{-1} \frac{I(u,v)}{R(u,v)}$ does not change if H is real

General Steps for Filtering

1. Multiply input image by $(-1)^{x+y}$ (centering)
2. Compute $F(u, v)$ (DFT)
3. Multiply $F(u, v)$ by $H(u, v)$ (filtering)
4. Compute inverse DFT of $H(u, v) \, F(u, v)$
5. Obtain the real part of the result
6. Mutliply by $(-1)^{x+y}$ (decentering)

From Spatial to Frequency Domain

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v)\, H(u, v)$$

$$\delta(x, y) * h(x, y) \Leftrightarrow \mathcal{F}[\delta(u, v)]\, H(u, v)$$

$$h(x, y) \Leftrightarrow H(u, v)$$

- Multiplication in the frequency domain is a convolution in the spatial domain.
- Given $h(x, y)$, we can obtain $H(u, v)$ by taking the inverse Fourier transform.

Padding (1)

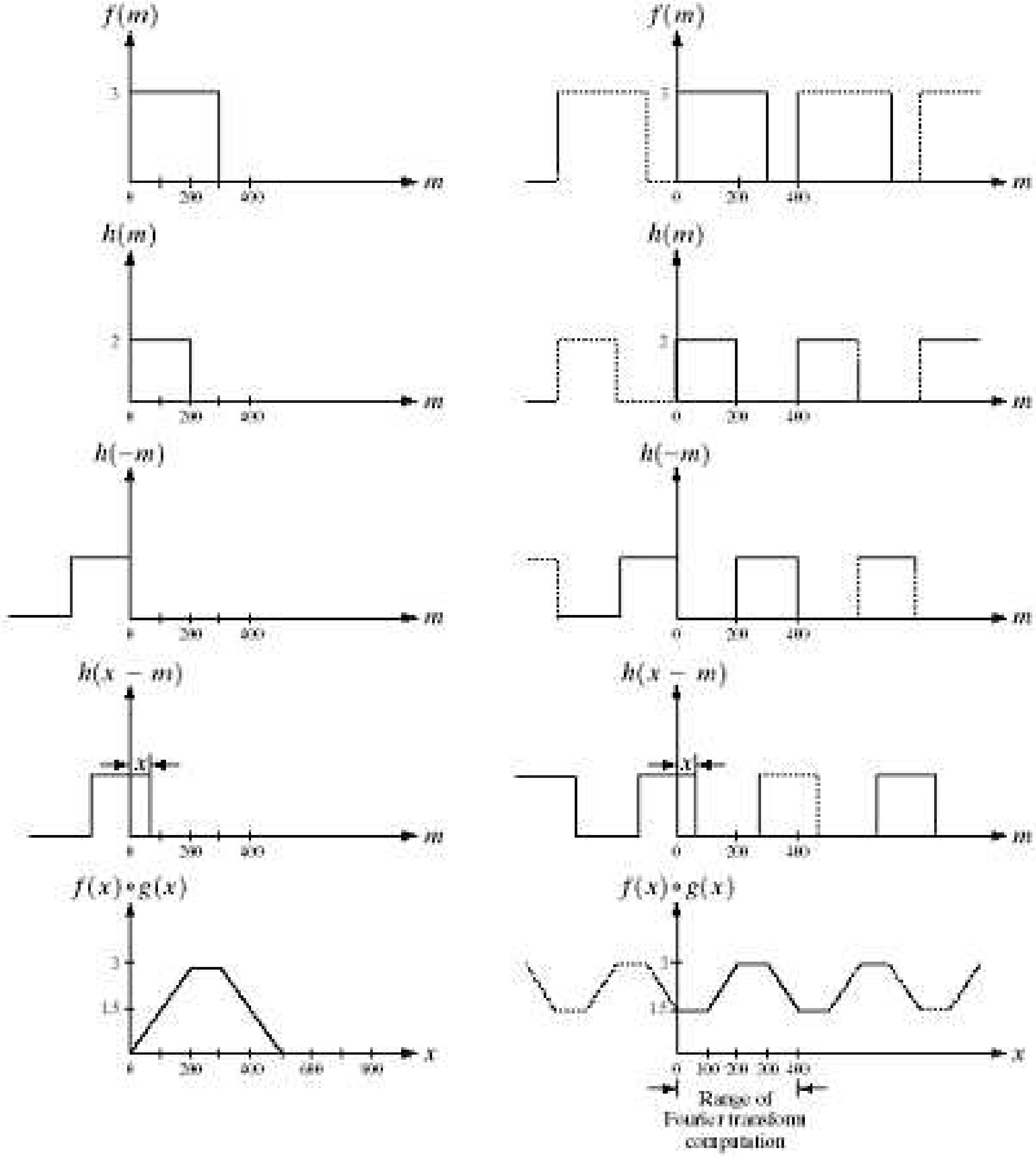
- convolution:

$$f(x) * h(x) = \frac{1}{M} \sum_{m=0}^{M-1} f(m)\, h(x-m)$$

- periodicity:

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

Padding (2)



a f
b g
c h
d i
e j

FIGURE 4.36 Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.

Padding (3)

Let P an identical period for f and g:

$$f_e(x) = \begin{cases} f(x) & 0 \leq x \leq A-1 \\ 0 & A \leq x \leq P \end{cases}$$

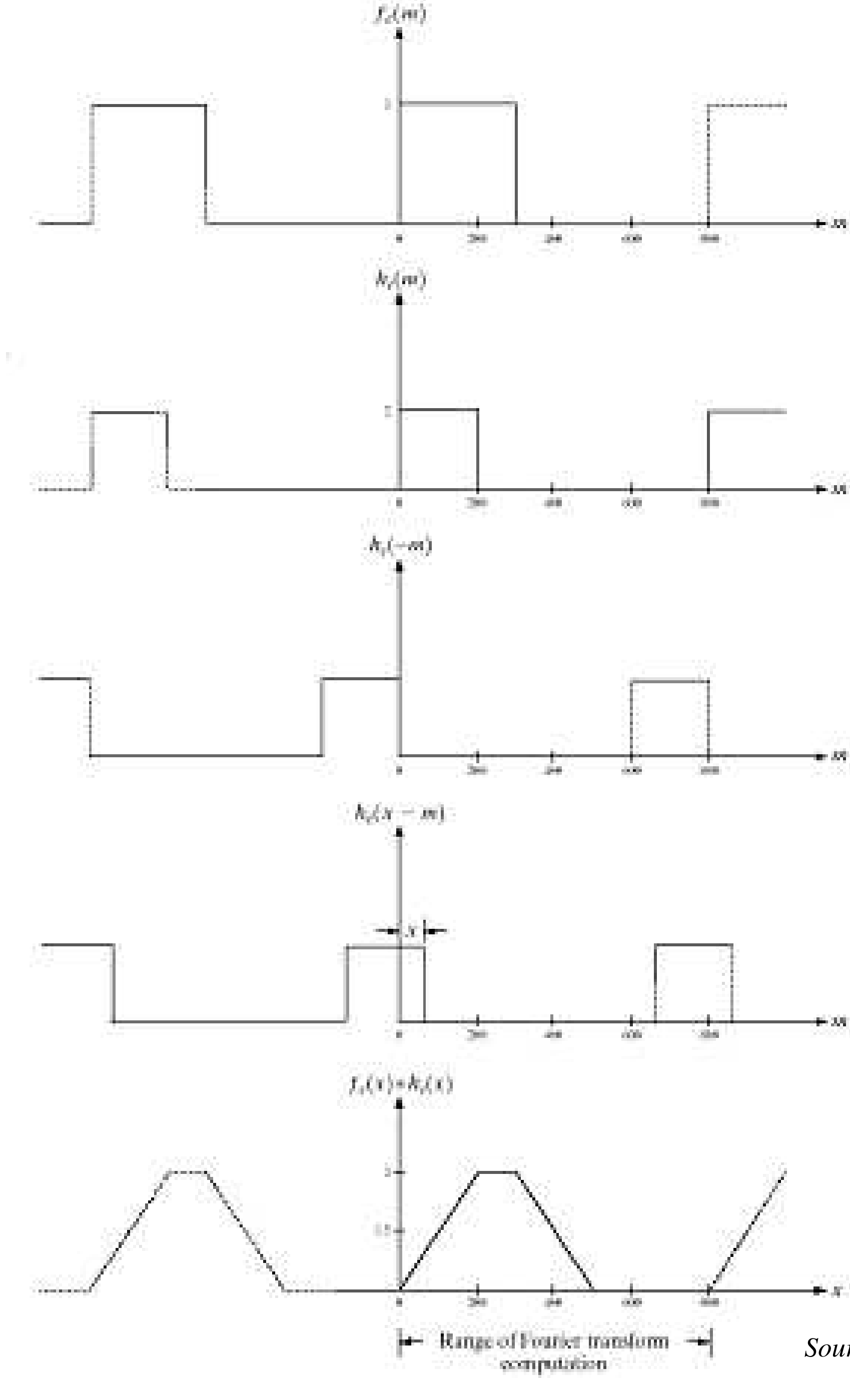
$$g_e(x) = \begin{cases} g(x) & 0 \leq x \leq B-1 \\ 0 & B \leq x \leq P \end{cases}$$

- If $P < A + B - 1$, the two signal will overlap: *wraparound error*.
- If $P > A + B - 1$, the periods will be separated.
- If $P = A + B - 1$, the periods will be adjacent.

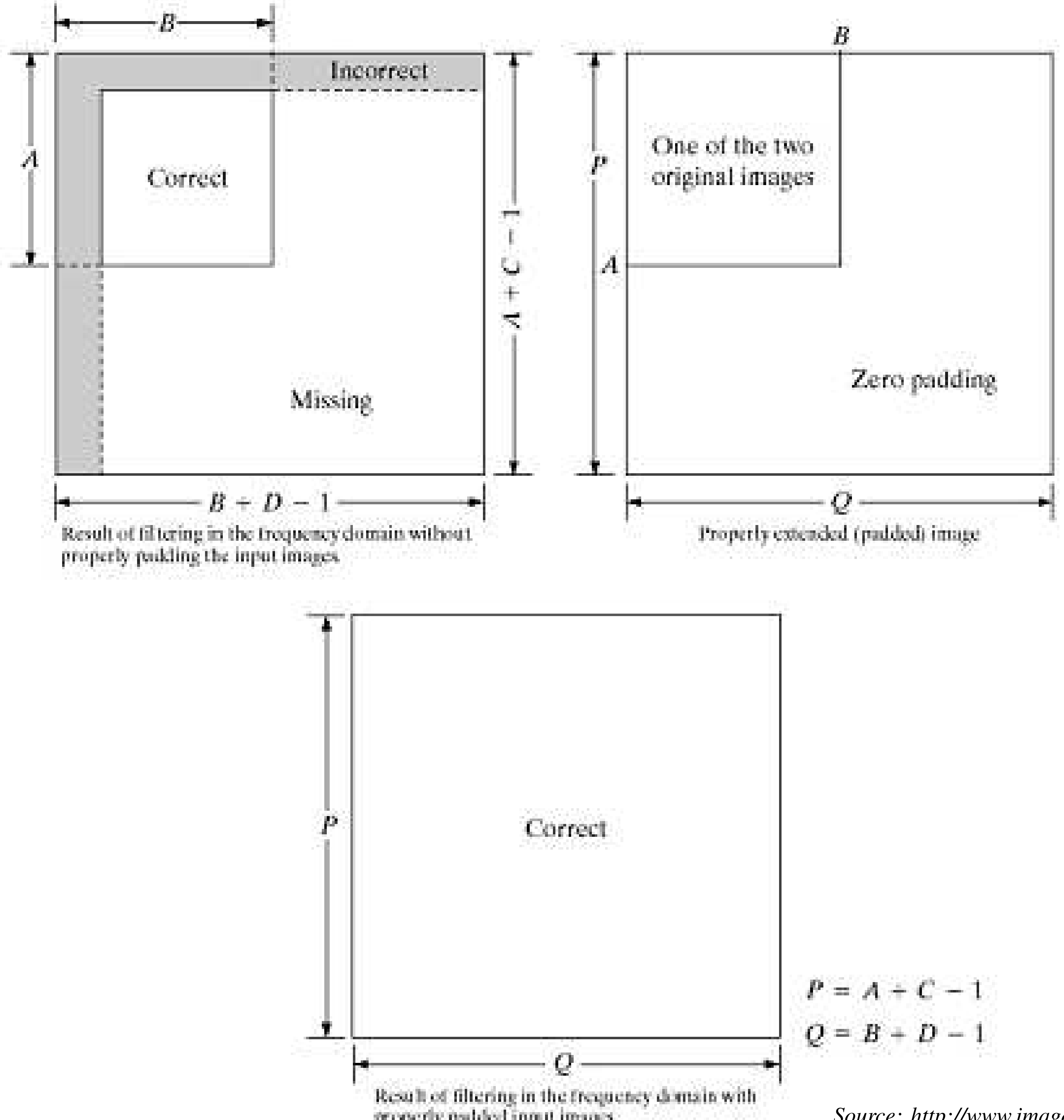
We can avoid *wraparound error* using $P \geq A + B - 1$.

In general, we use $P = A + B - 1$.

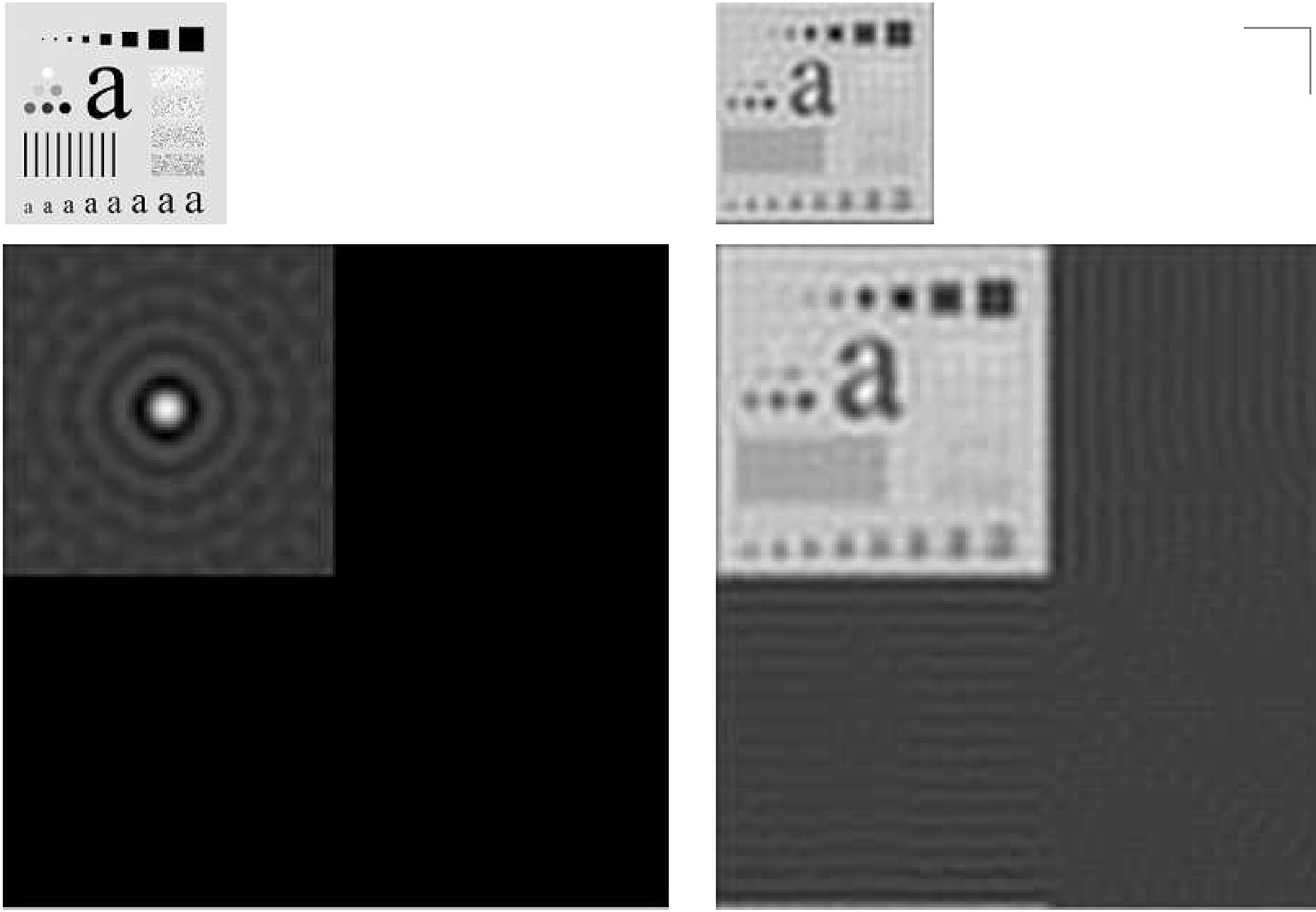
Padding (4)



Padding (5)

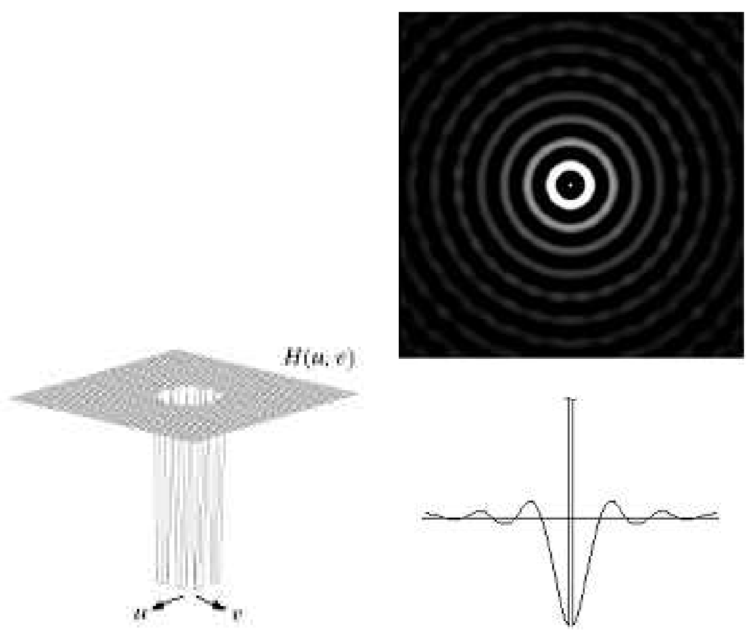

B
Incorrect
A
Correct
A + C − 1
Missing
B + D − 1
Result of filtering in the frequency domain without properly padding the input images
B
P
One of the two original images
A
Zero padding
Q
Properly extended (padded) image
P
Correct
Q
P = A + C − 1
Q = B + D − 1
Result of filtering in the frequency domain with properly padded input images.

Padding: Example



Spatial Representation of a Filter

1. Multiply filter $H(u,v)$ by $(-1)^{u+v}$ (centering)
2. Compute the inverse DFT
3. Multiply the real part of the inverse DFT by $(-1)^{x+y}$

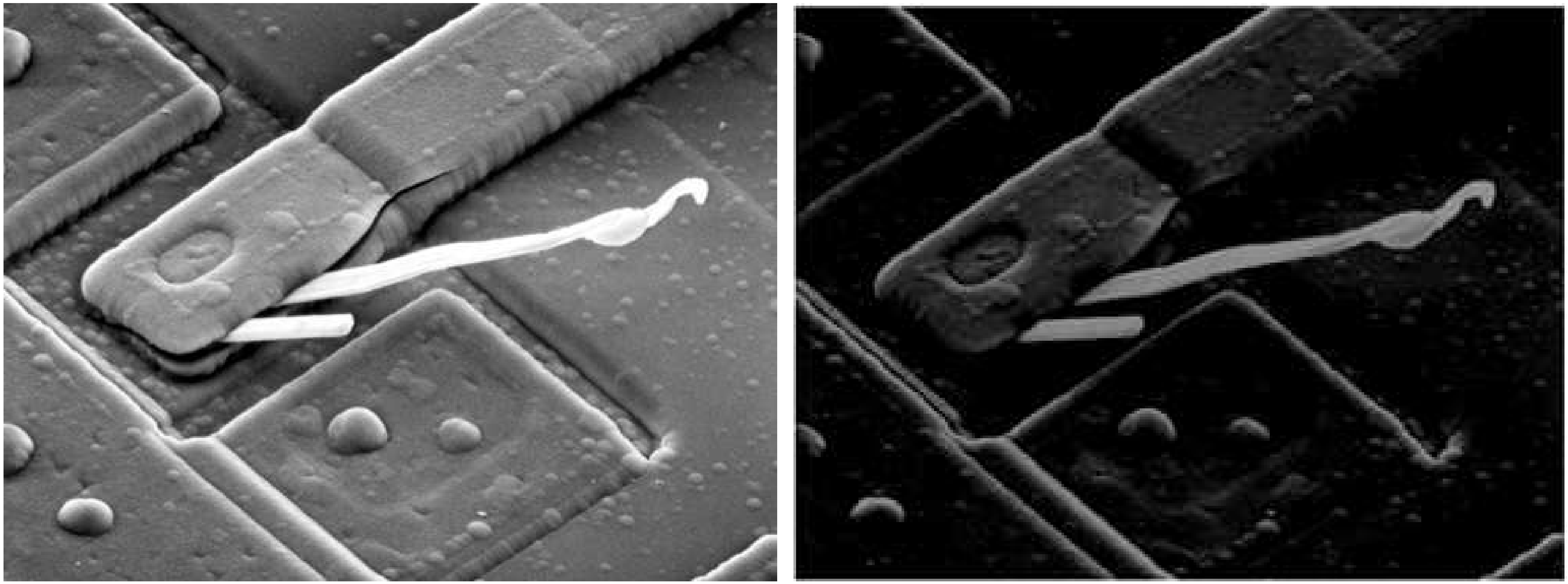


A Simple Filter: Notch Filter (1)

- We wish to force the average value of an image to zero:
  - $F(0,0)$ is the average value of the image
  - if size of the image is $M \times N$ then the centered value of the Fourier transform is the average value $(\frac{M}{2}, \frac{N}{2})$

$$H(u,v) = \begin{cases} 0 \text{ if } (u,v) = (\frac{M}{2}, \frac{N}{2}) \\ 1 \text{ otherwise.} \end{cases}$$

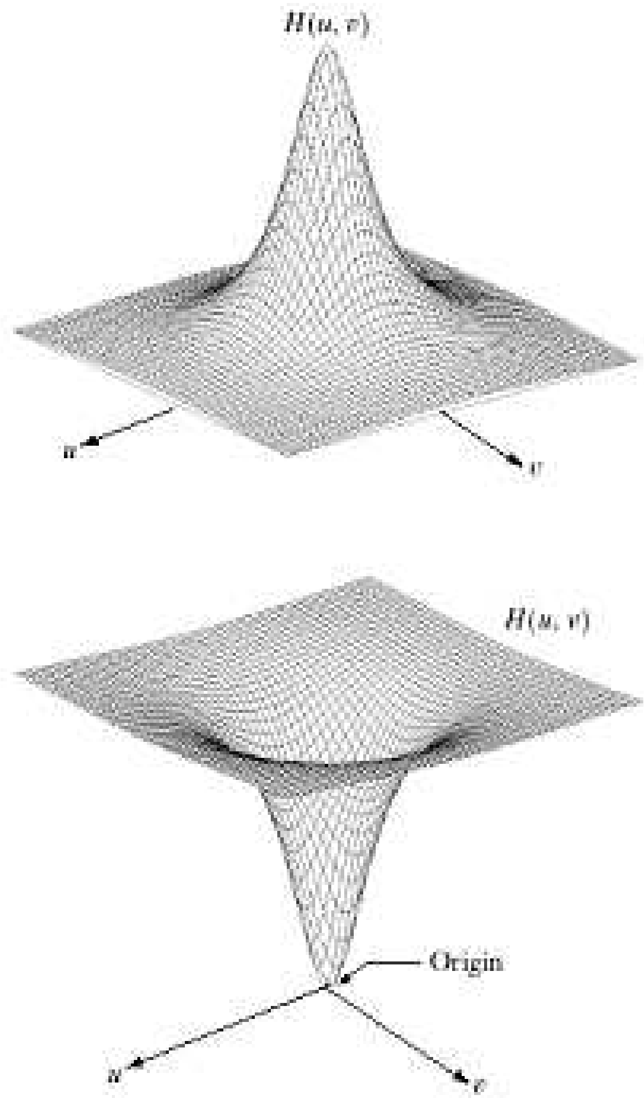
A Simple Filter: Notch Filter (2)

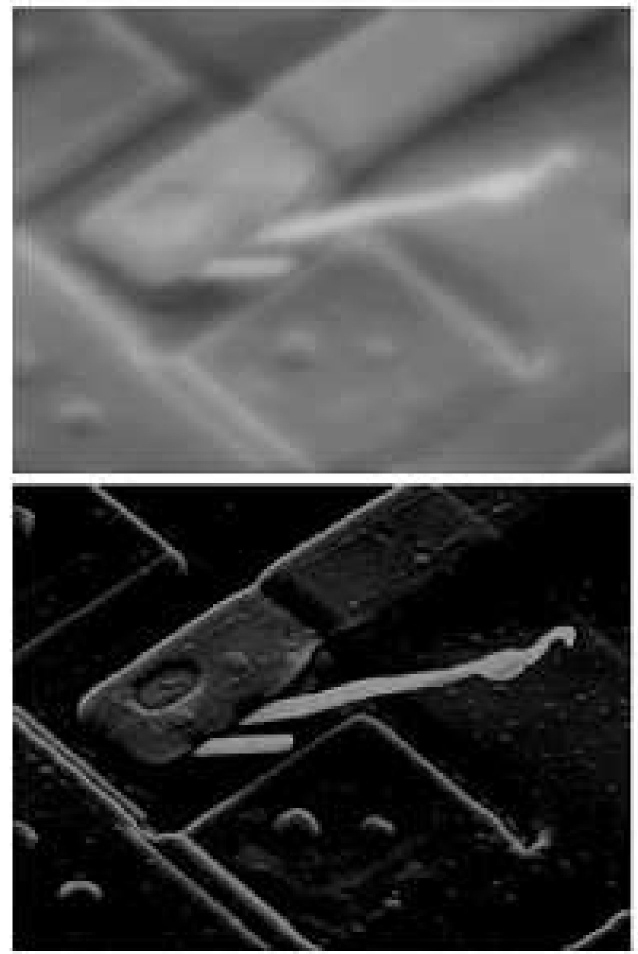


notch filter (constant function with a hole at the origin)

$$H(u,v) = \begin{cases} 0 \text{ if } (u,v) = (\frac{M}{2}, \frac{N}{2}) \\ 1 \text{ otherwise.} \end{cases}$$

Lowpass and Highpass Filter





Smoothing Frequency-Domain Filters

$$G(u,v) = H(u,v)\,F(u,v)$$

- Smoothing: attenuating specified range of high-frequency components
- Three types of *lowpass filter*:
  - ideal (very sharp)
  - Butterworth (tunable)
  - Gaussian (very smooth)

Ideal Lowpass Filters (1)

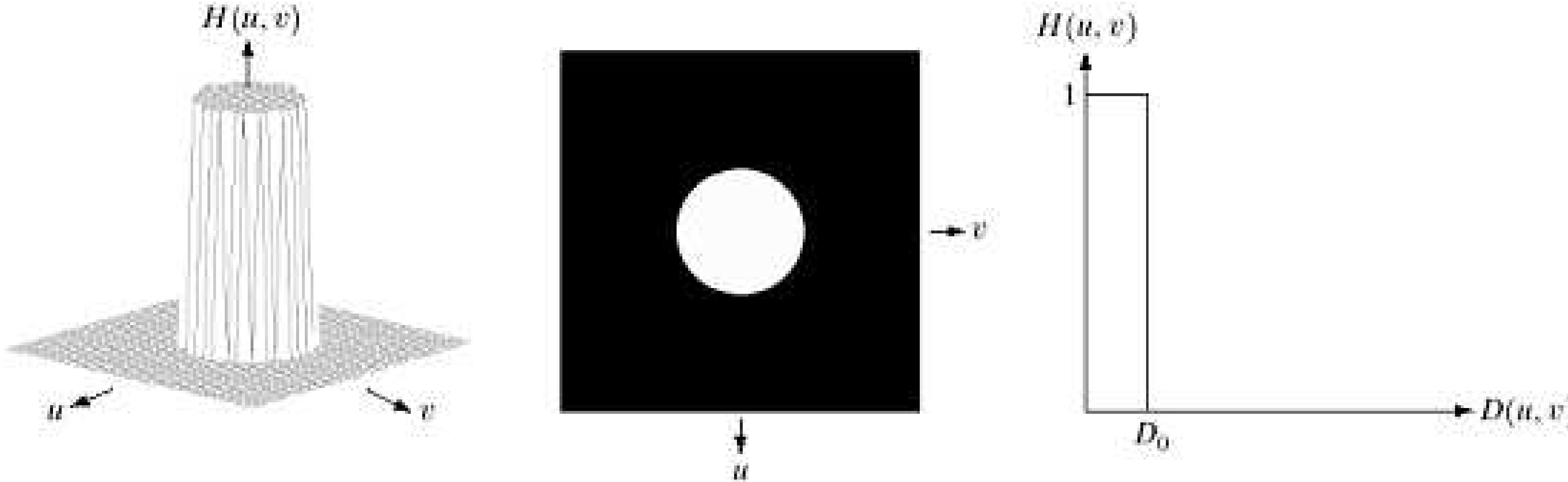
$$H(u,v) = \begin{cases} 1 \text{ if } D(u,v) \leq D_0 \\ 0 \text{ if } D(u,v) > D_0 \end{cases}$$

- D_0: nonnegative quantity
- $D(u,v)$: distance from a point (u,v) to the origin
- origin: $(\frac{M}{2}, \frac{N}{2})$ (centered)

Then,

$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

Ideal Lowpass Filters (2)

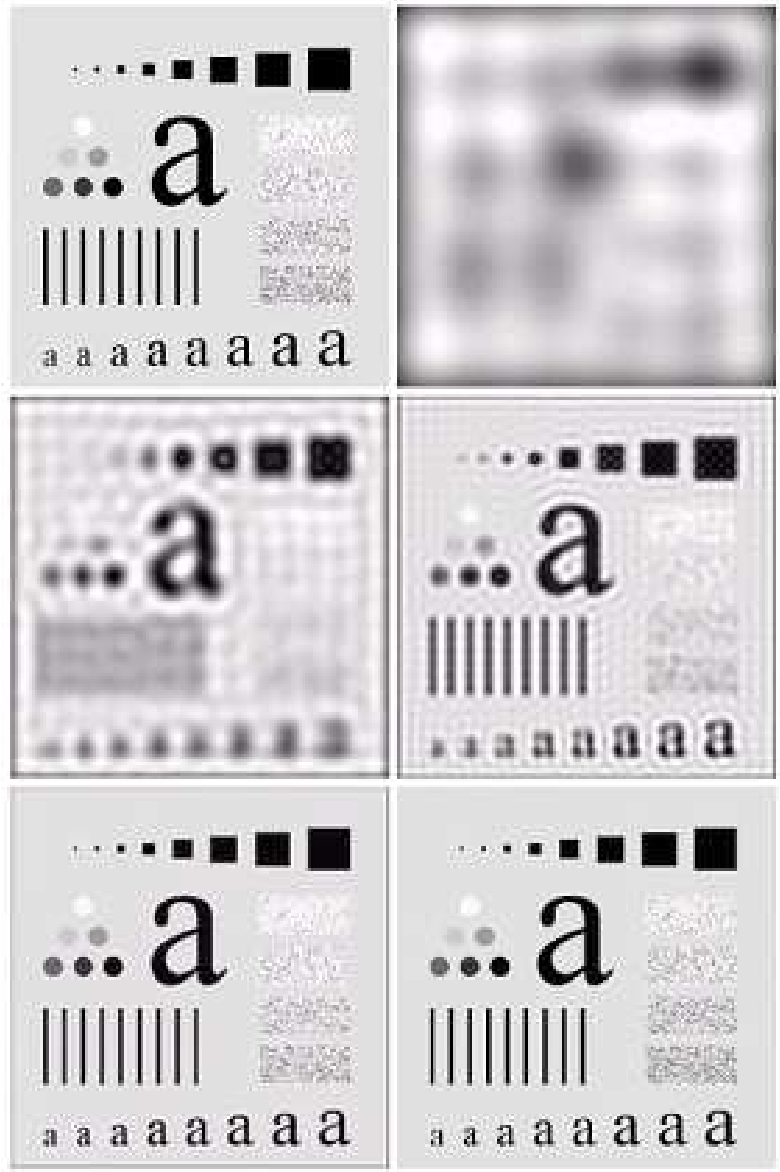


- ideal filter (ILPF): all frequencies inside a circle of radius D_0 are passed with no attenuation
- In our case:
  - zero-phase-shift filter
  - radially symmetric
- Transition: *cutoff frequency*
- Ringing behavior of the filter

Ideal Lowpass Filters: Example

10 (13%)

30 (6.9%)



60 (6.9%)

160 (4.3%)

160 (2.2%)

460 (0.8%)

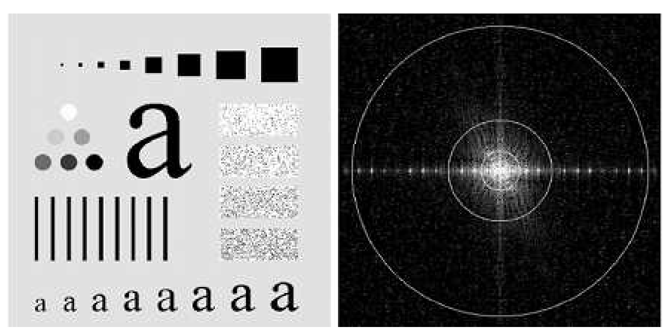
Cutoff Frequency

- Compute circles that enclose specific amount of total image power P_T

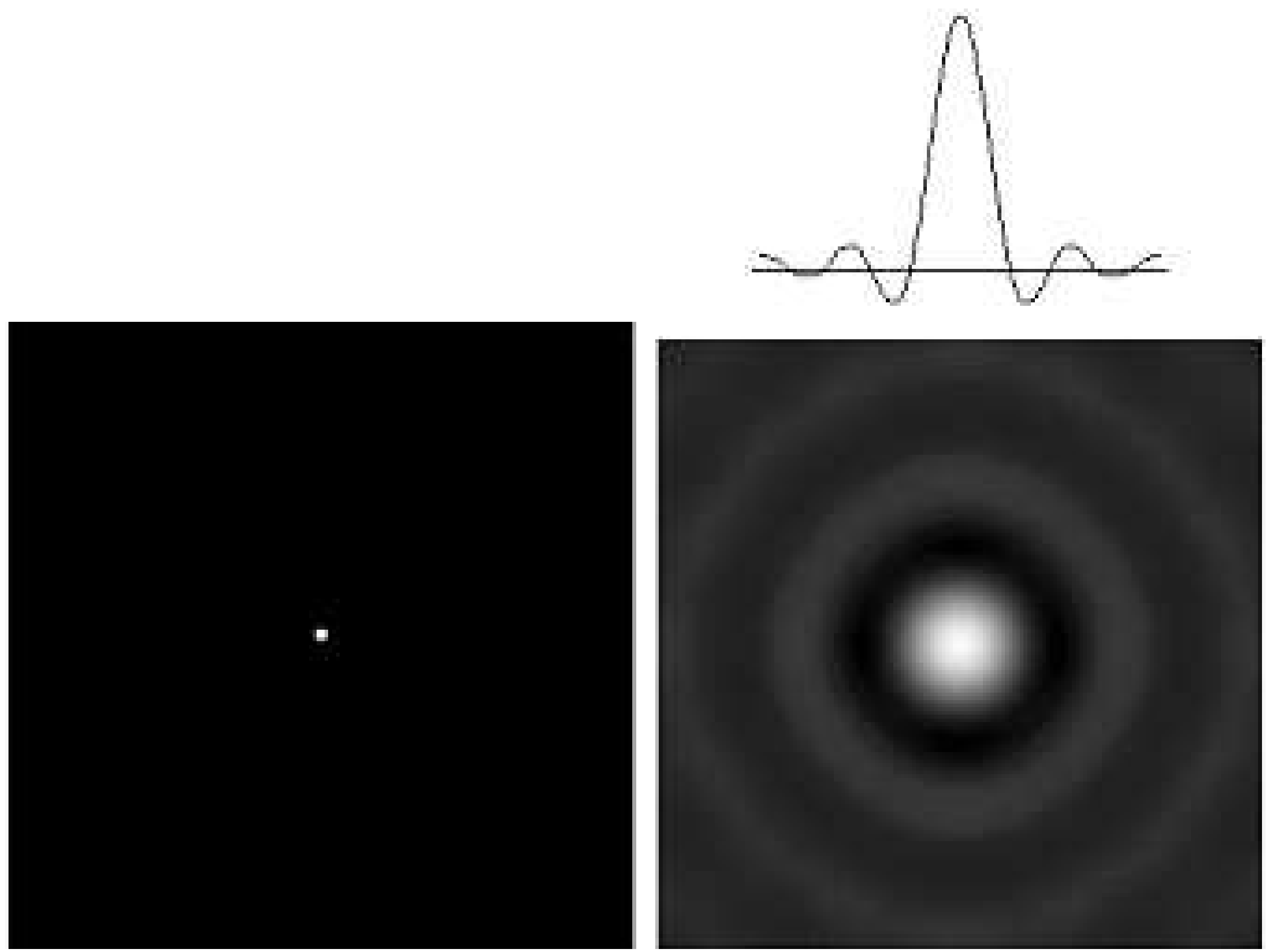
$$P_T = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u,v)$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

$$\alpha = \frac{100}{P_T} \sum_u \sum_v P(u,v)$$



Ringing Effect of ILPF



Butterworth Lowpass Filters

- BLPF of order n, with a cutoff frequency distance D_0 is defined as

$$H(u,v) = \frac{1}{1 + [\frac{D(u,v)}{D_0}]^{2n}}$$

$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

- no clear cutoff between passed and filtered frequencies

Butterworth Lowpass Filters (2)

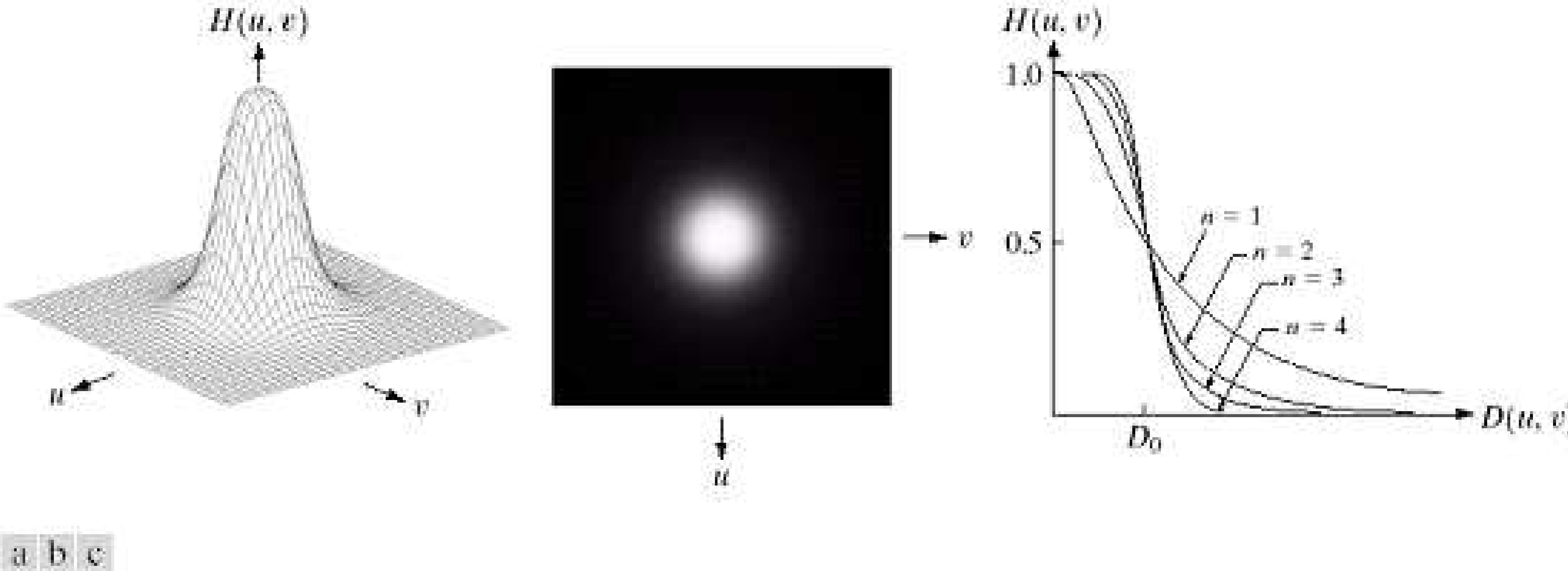
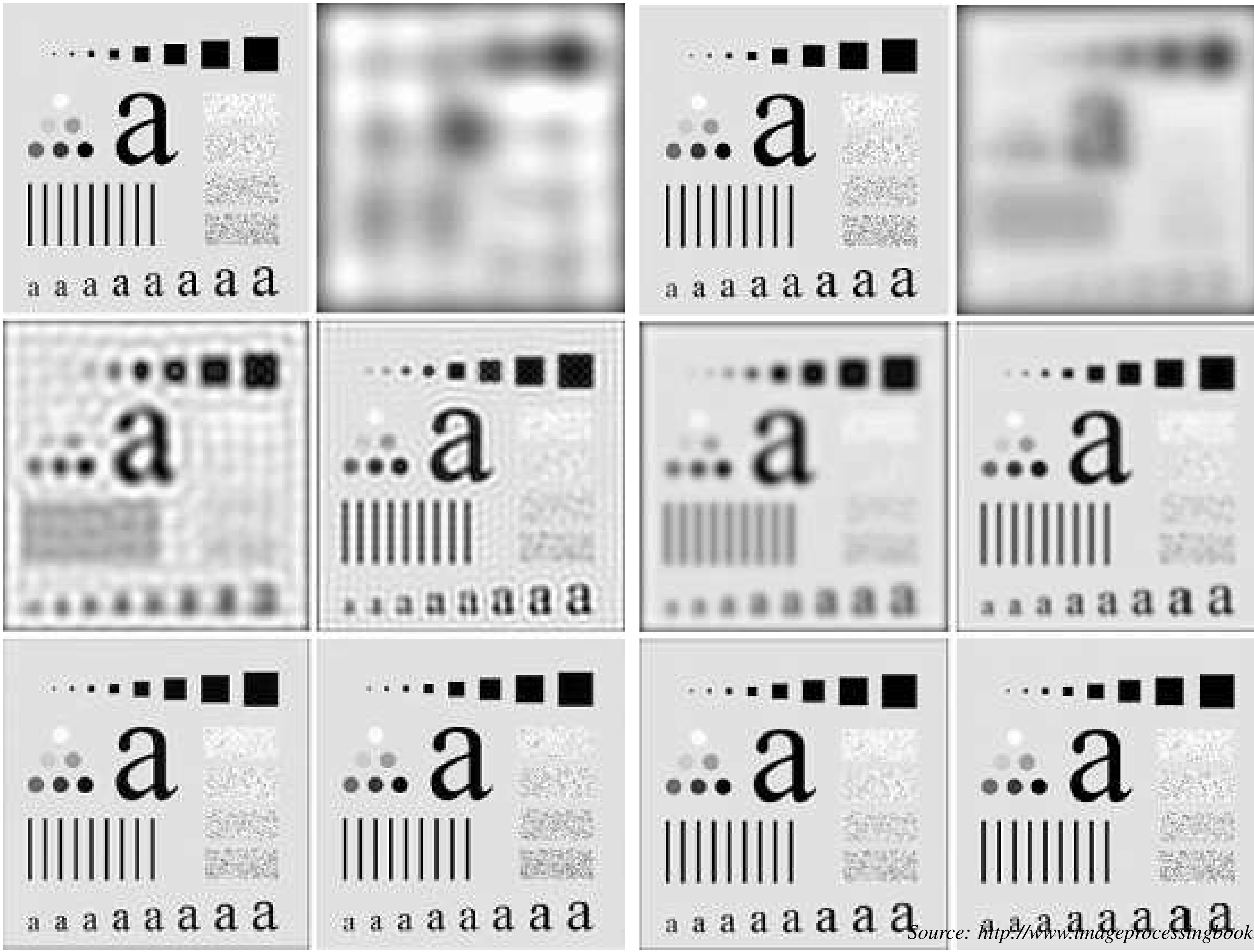


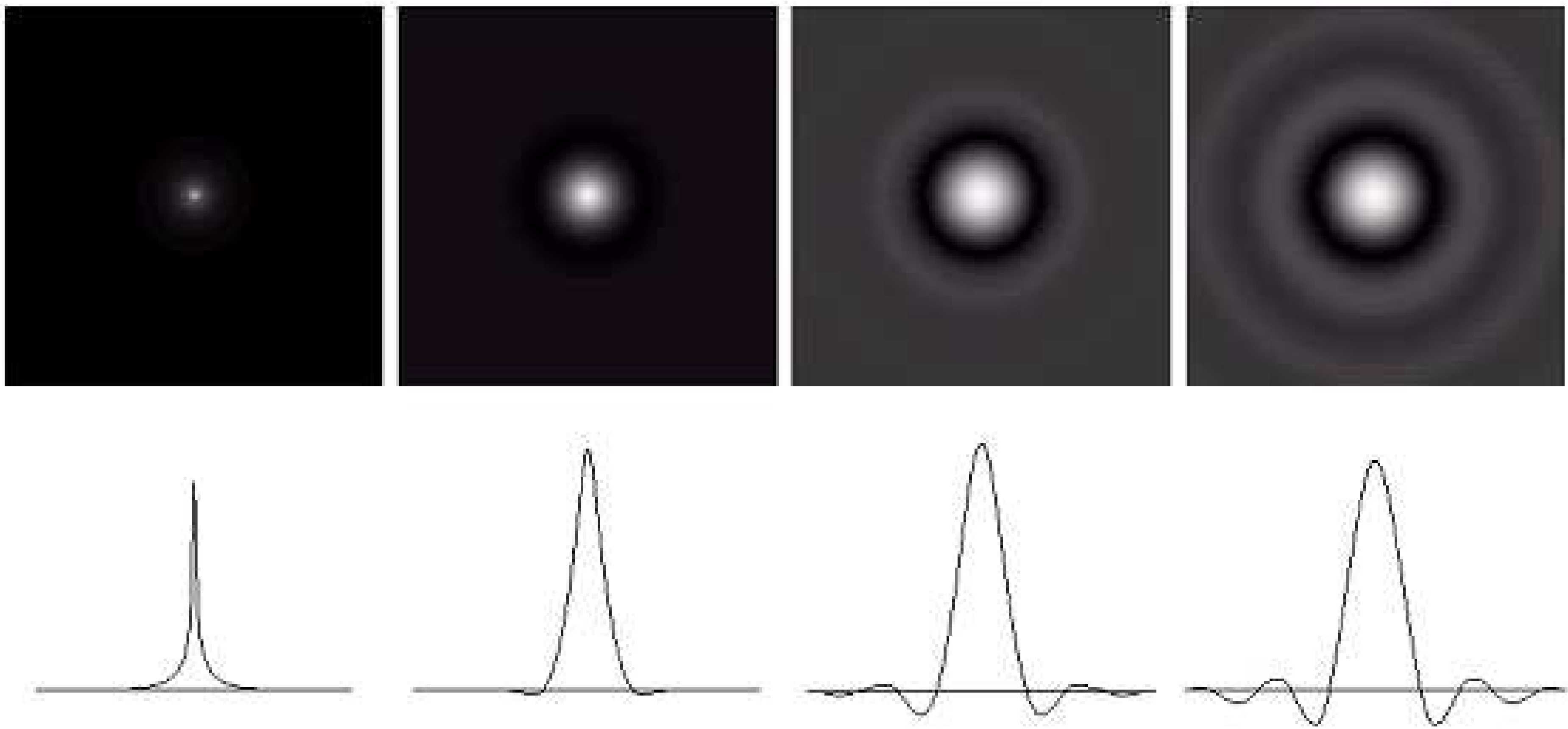
FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Lowpass Filters: Example

- $n = 2$, radii=$5, 15, 30, 80, 230$



Ringing Effect of BLPF



a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Lowpass Filters (1)

- The form of a gaussian lowpass filter GLPF in 2D is:

$$H(u,v) = e^{\frac{-D^2(u,v)}{2\sigma^2}}$$

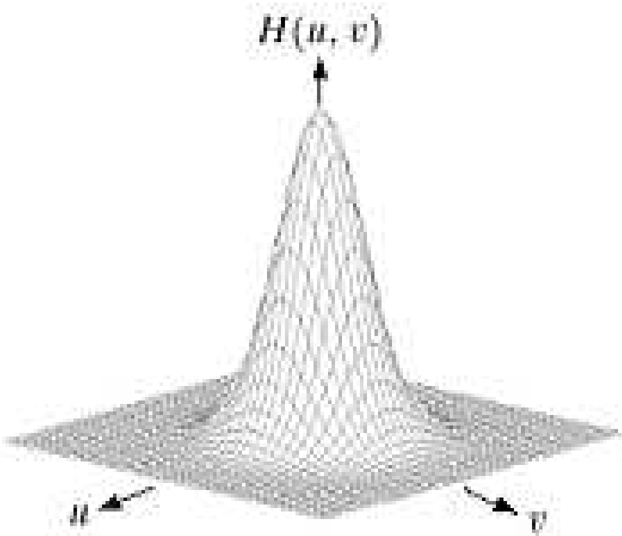
$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

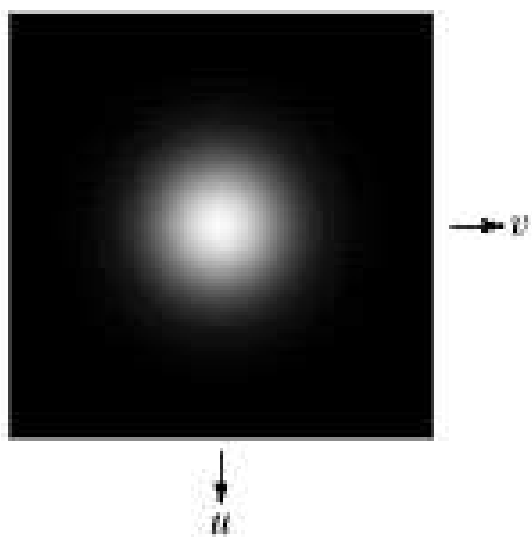
- The inverse Fourier transform of a GLPF is also a Gaussian
- A spatial Gaussian filter will have no ringing

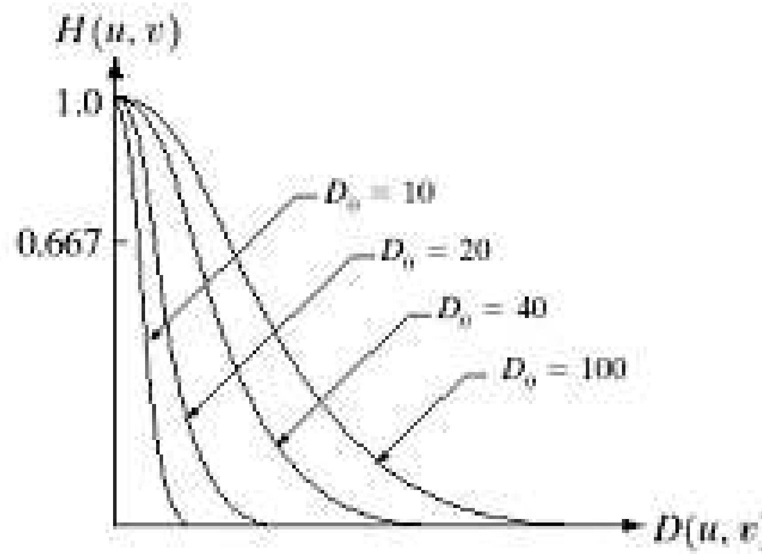
Gaussian Lowpass Filters (2)

- σ: mesure of the spread of the Gaussian curve
- Let $\sigma = D_0$, then:

$$H(u,v) = e^{\frac{-D^2(u,v)}{2D_0^2}}$$

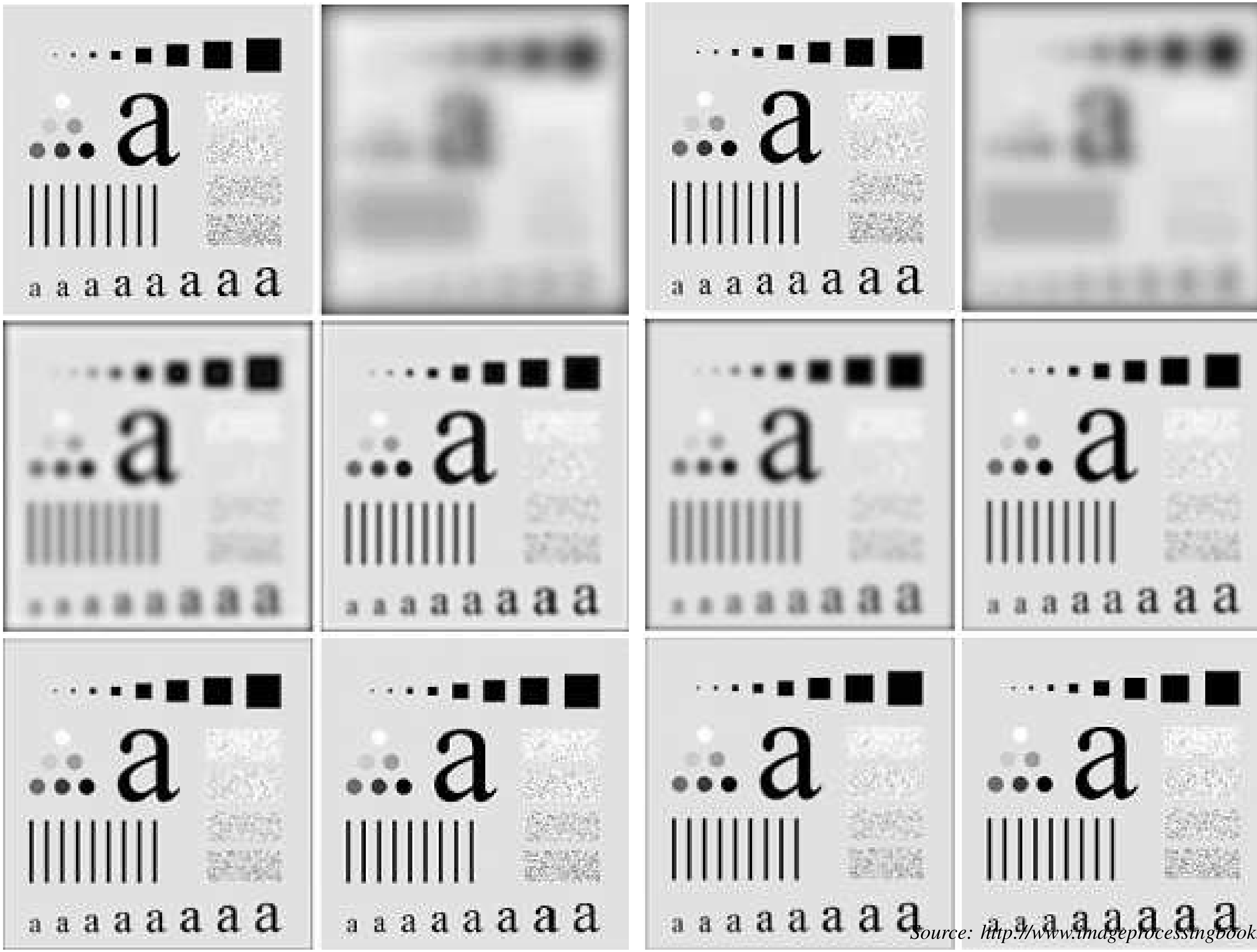




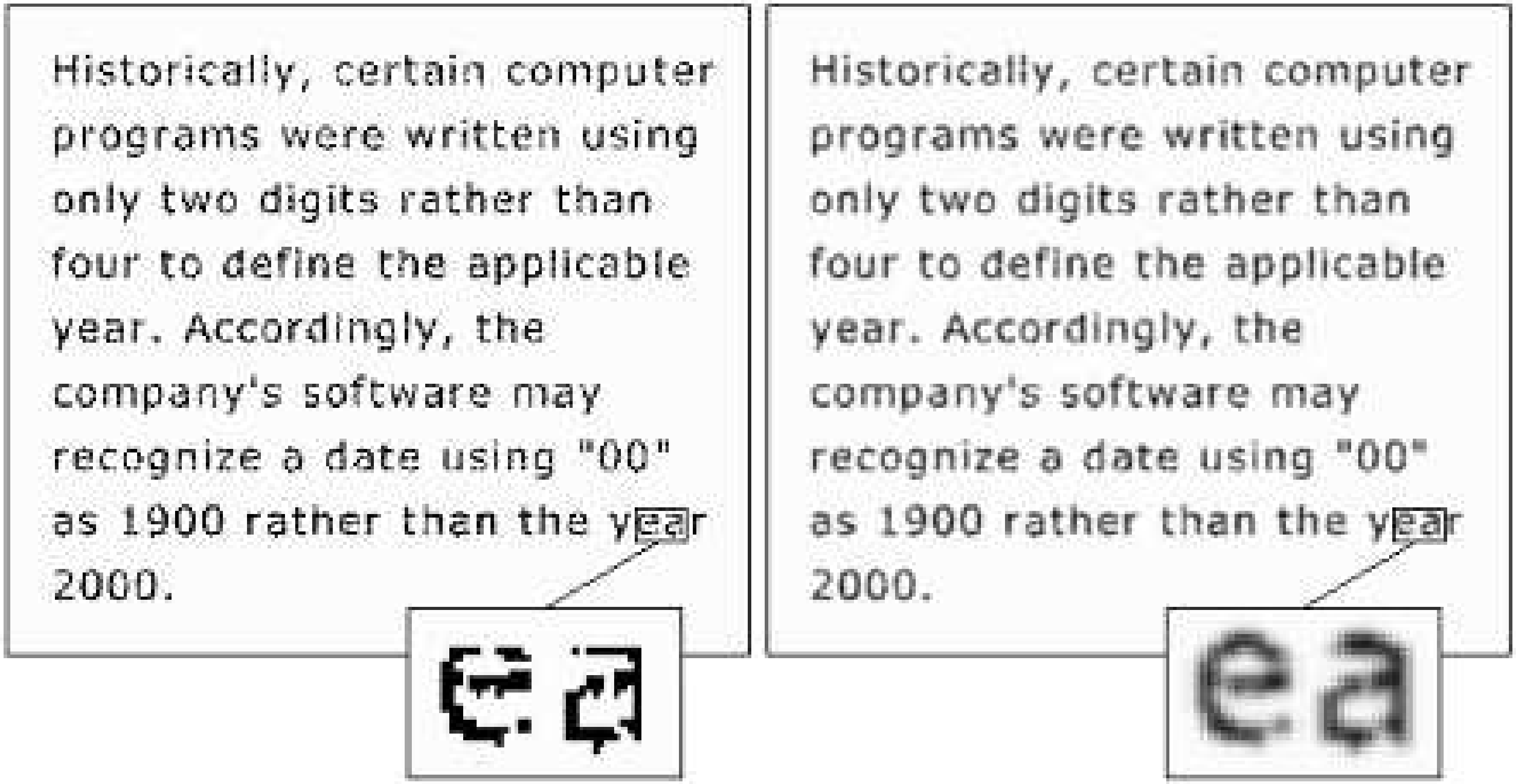


Gaussian Lowpass Filters: Example (1)

- $n = 2$, radii=$5, 15, 30, 80, 230$



Gaussian Lowpass Filters: Example (2)



Highpass Filters (1)

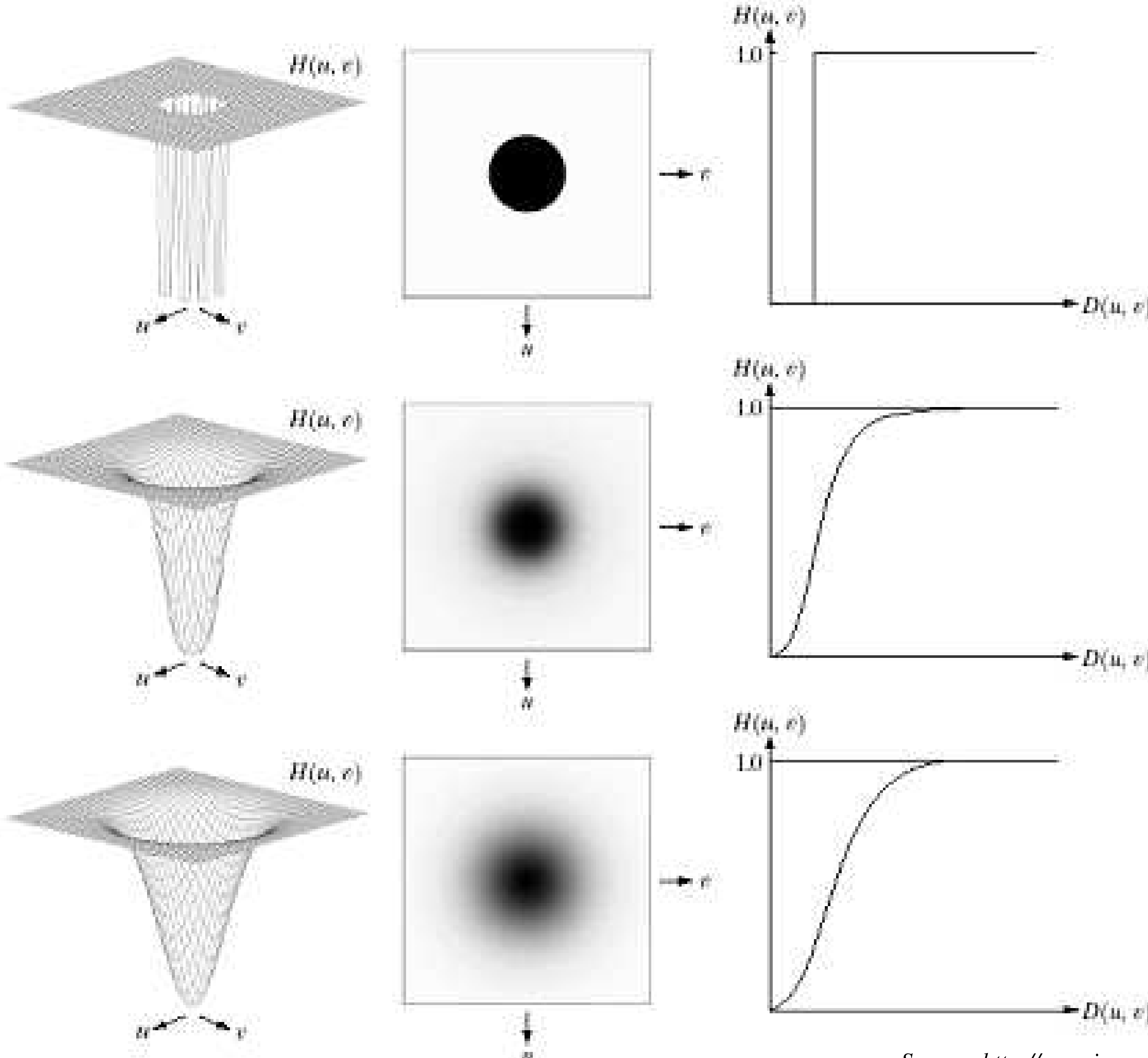
- Highpass filter: image sharpening (low-frequency attenuation)
- In our case:
  - zero-phase-shift filter
  - radially symmetric

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

with:

- $H_{lp}(u,v)$: transfer function of the corresponding lowpass filter
- $H_{hp}(u,v)$: transfer function of the corresponding highpass filter

Highpass Filters (2)



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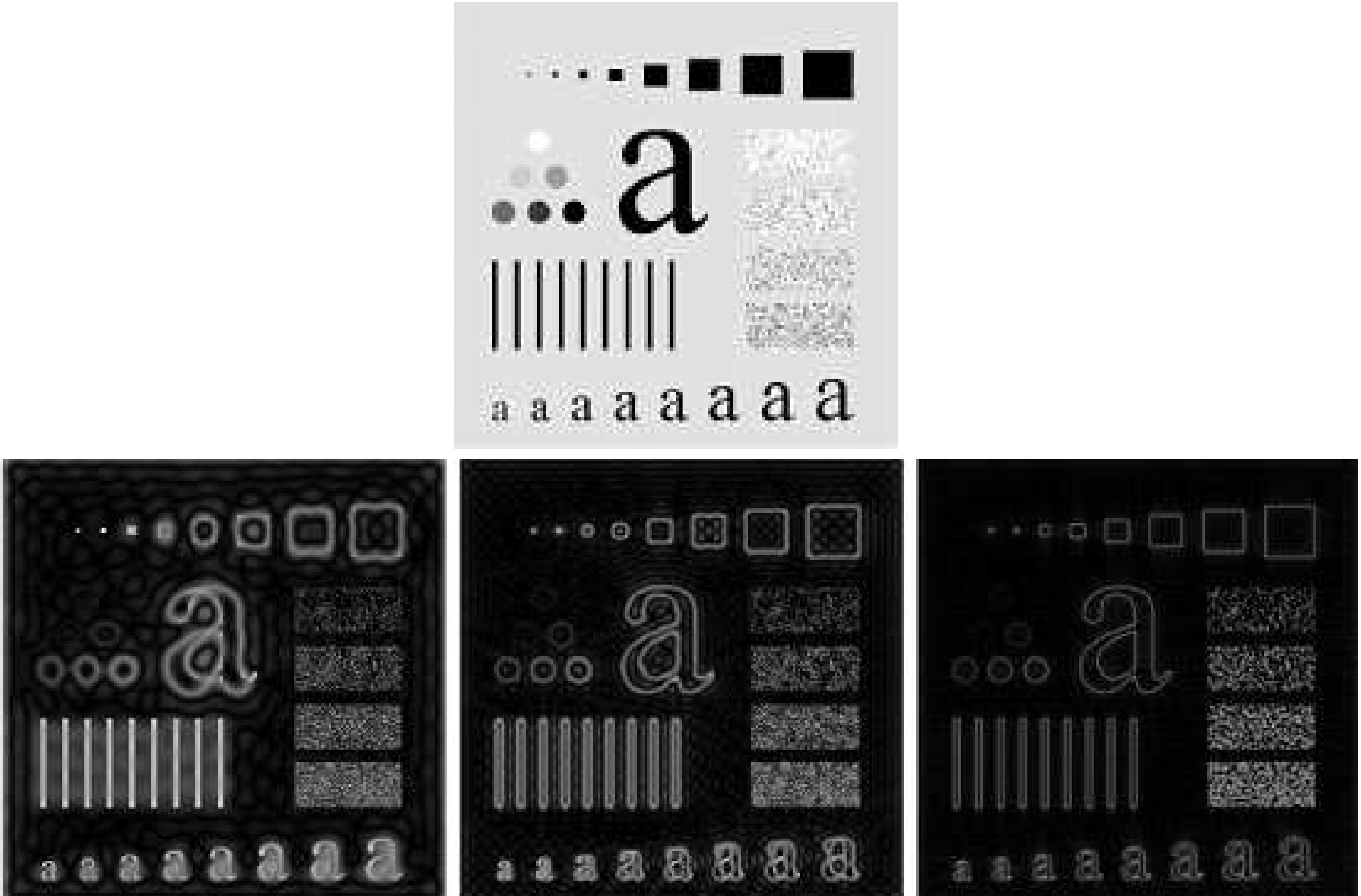
- D_0: nonnegative quantity
- $D(u,v)$: distance from a point (u,v) to the origin
- origin: $(\frac{M}{2}, \frac{N}{2})$ (centered)

Then,

$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

Ideal Lowpass Filters: example



Butterworth Highpass Filters

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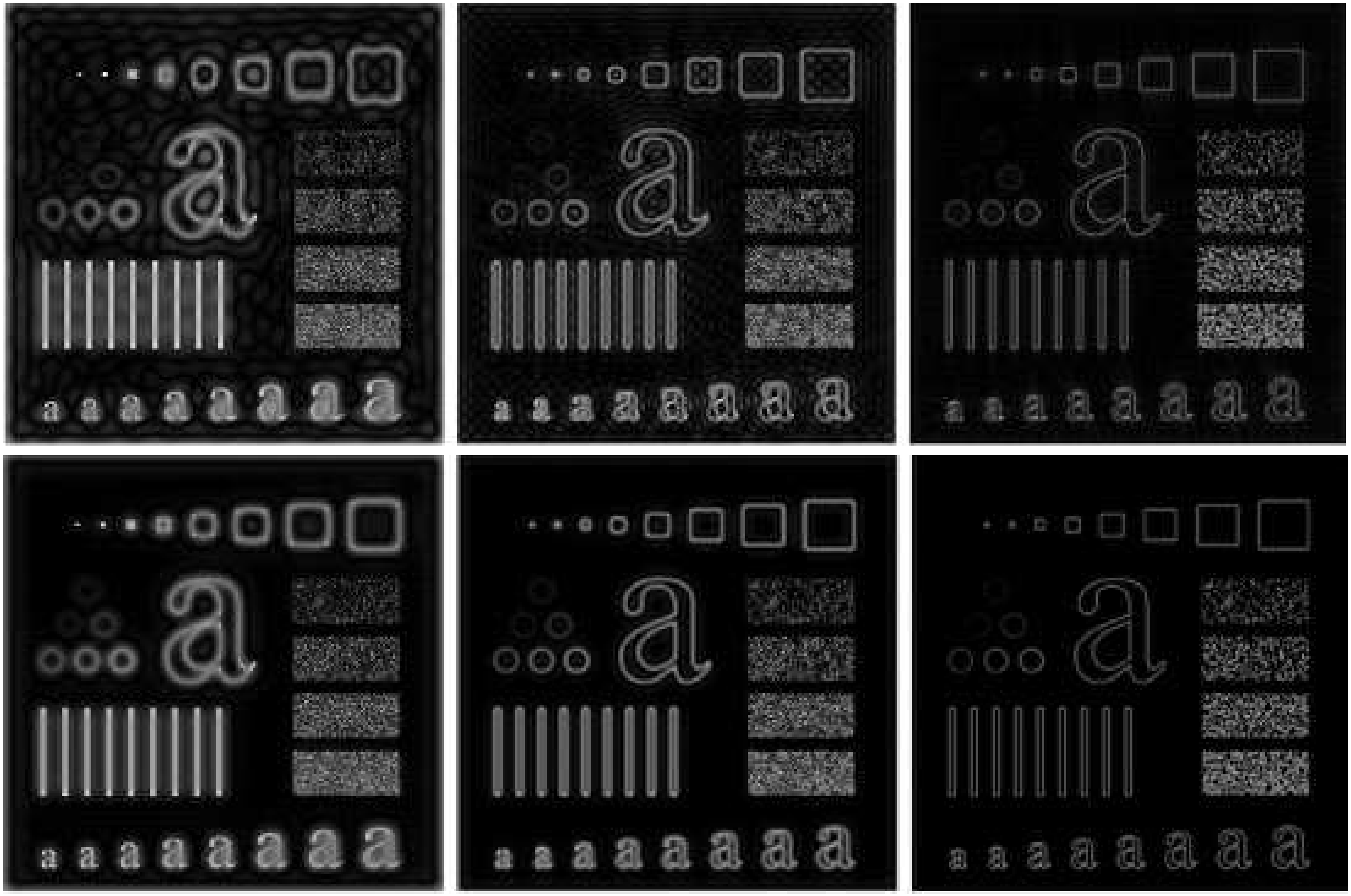
$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

- no clear cutoff between passed and filtered frequencies

Butterworth Highpass Filters: Example



Gaussian Highpass Filters

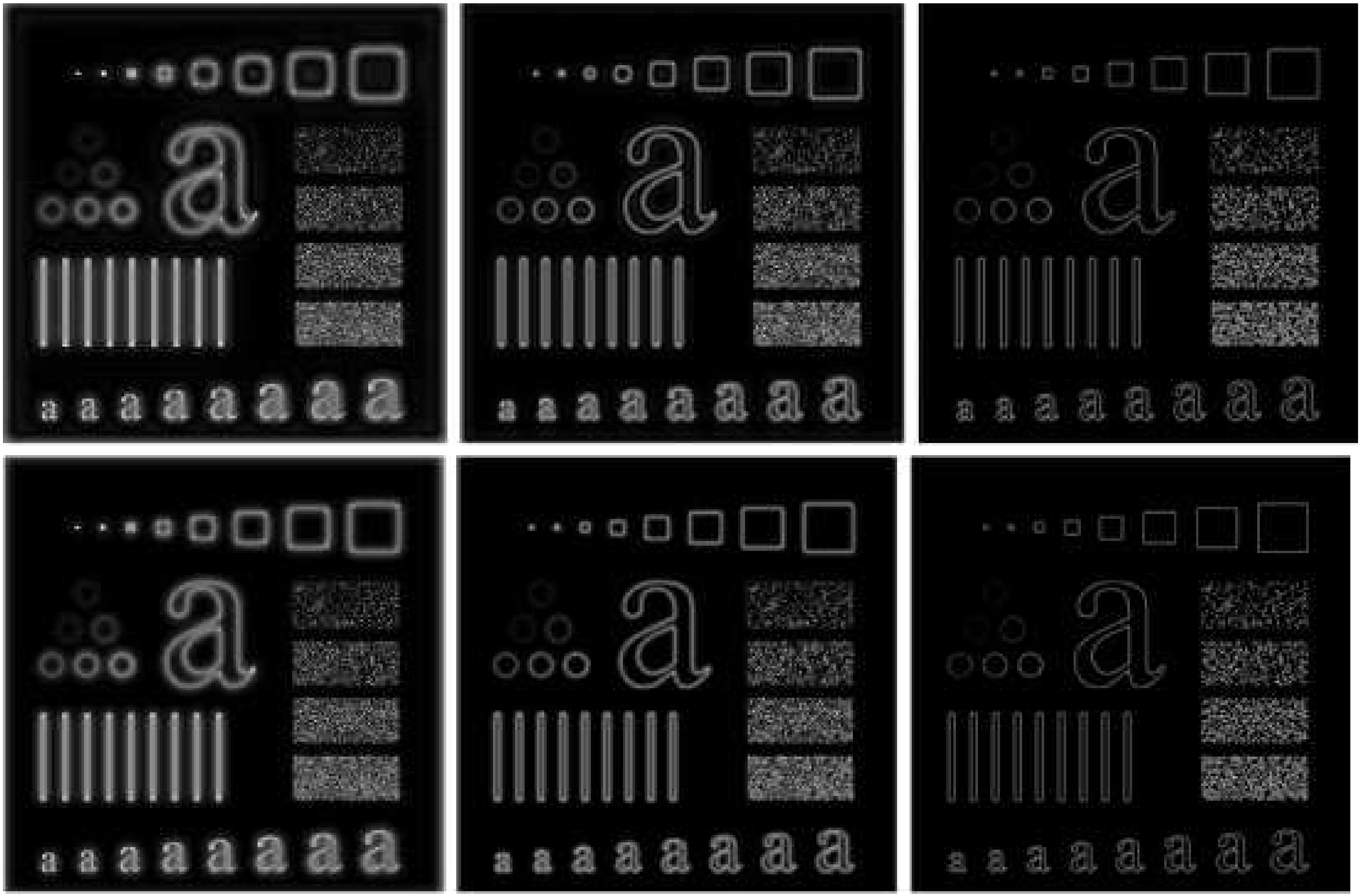
- The form of a gaussian lowpass filter GLPF in 2D is:

$$H(u,v) = 1 - e^{\frac{-D^2(u,v)}{2D_0^2}}$$

$$D(u,v) = \sqrt{(u - \frac{M}{2})^2 + (v - \frac{N}{2})^2}$$

- The inverse Fourier transform of a GLPF is also a Gaussian
- A spatial Gaussian filter will have no ringing

Gaussian Highpass Filters: Example



Laplacian: Spatial Domain (1)

- $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
  - $\frac{\partial^2 f}{\partial x^2} = f(x+1,y) + f(x-1,y) - 2f(x,y)$
  - $\frac{\partial^2 f}{\partial y^2} = f(x,y+1) + f(x,y-1) - 2f(x,y)$
- $\nabla^2 f = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)$

0	1	0
1	−4	1
0	1	0

1	1	1
1	−8	1
1	1	1

Laplacian: Spatial Domain (2)

$$g(x,y) = \begin{cases} f(x,y) - \nabla^2 f(x,y) & \text{if the center coefficient is negative} \\ f(x,y) + \nabla^2 f(x,y) & \text{if the center coefficient is positive} \end{cases}$$

Laplacian: Frequency Domain (1)

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$

Then,

$$\begin{aligned}\mathcal{F}\left[\tfrac{\partial^2 f(x,y)}{\partial x^2} + \tfrac{\partial^2 f(x,y)}{\partial y^2}\right] &= (ju)^2 F(u,v) + (jv)^2 F(u,v) \\ &= -(u^2 + v^2) F(u,v)\end{aligned}$$

Finally,

$$\mathcal{F}[\nabla^2 f(x,y)] = -(u^2 + v^2) F(u,v)$$

The Laplacian filter in the frequency domain is:

$$H(u,v) = -(u^2 + v^2)$$

Laplacian: Frequency Domain (2)

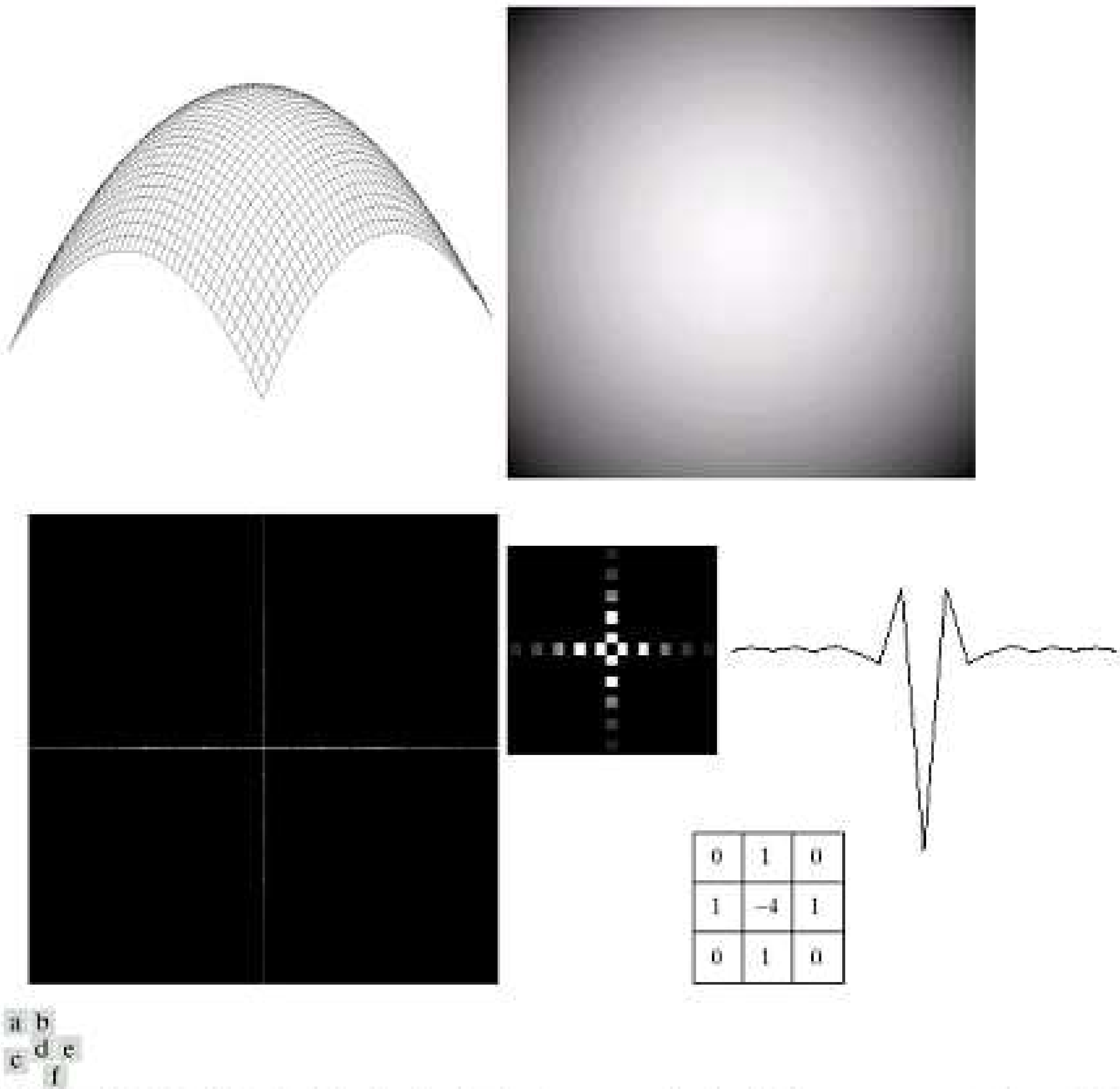


FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

Laplacian: Example