Mathematical morphology: basic operators

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Graph-based mathematical morphology

2022

Outline of the lecture

Operator

- E is a set
- Let $X \subseteq E$, we denote by $\overline{X}$, the *complementary set of X*
  - $\overline{X} = E \setminus X$
  - We remark that $\overline{(\overline{X})} = X$

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Definition

- *An operator (on E) is a map from $\mathcal{P}(E)$ to $\mathcal{P}(E)$*
- *In the following γ denotes an operator on E*

Dual

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- *The* dual *of* γ *is the operator* $\star\gamma$ *defined by*
  - $\forall X \subseteq E$, $\star\gamma(X) = \overline{\gamma(\overline{X})}$

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Property

- $\star\star\gamma = \gamma$

Dual: Example #1

- Let E be a metric space: E is endowed with a given distance d
- Let γ^r be the operator defined by
  - $\forall X \in \mathcal{P}(E)$, $\gamma^r(X) = \{x \in E \mid \exists y \in X, d(x,y) \leq r\}$

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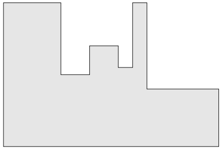
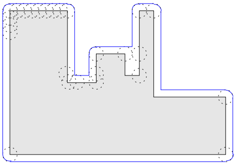


X



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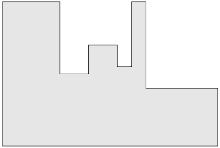

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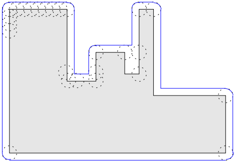

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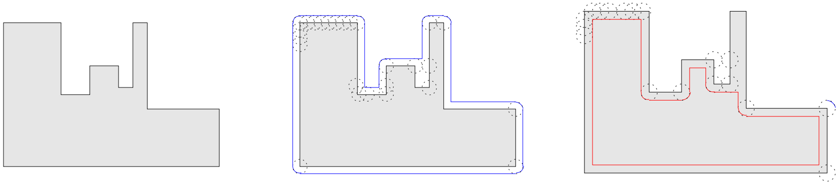

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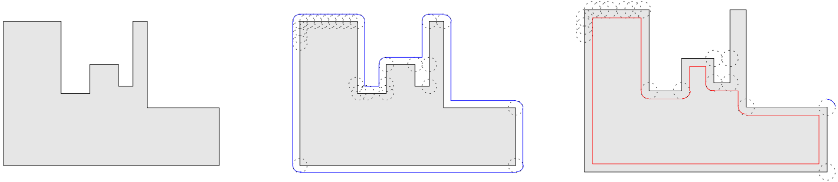
- The set $\gamma^r(X)$ can be seen as a neighborhood of X (of size r) and $\star\gamma(X)$ as an interior of X



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Dual: Exemple #2 - convex hull

- Let E be the Euclidean plan $E = \mathbb{R}^2$
- A subset Y of E is convex if any line segment whose extremities are in Y is included in Y
- Let $X \subseteq E$ the *convex hull of X* is the set
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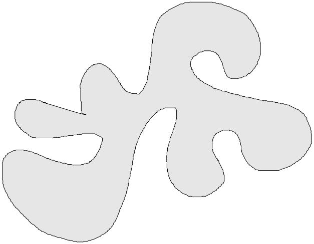

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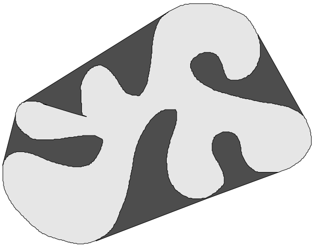

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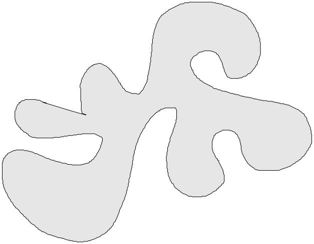

Property

- *Let $X \subseteq E$ be a bounded set, then $\star ch(X) = \emptyset$.*
- *Is the converse also true?*

NB: X is bounded $\Leftrightarrow$ $\exists$ a disc of finite radius that contains X

Extensive operator

Definition

- *An operator* γ *is* extensive *if*
  - $\forall X \subseteq E,\ X \subseteq \gamma(X)$
- *An operator* γ *is* anti-extensive *if*
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Property

- *An operator γ is extensive if and only if $\star\gamma$ is anti-extensive*

Proof. γ is extensive $\Leftrightarrow \forall X \subseteq E, \overline{X} \subseteq \gamma(\overline{X})$

Thus, $\forall X \subseteq E, X \supseteq \overline{\gamma(\overline{X})}$, which means that $\star\gamma$ is anti-extensive. □

Increasing and idempotent operators

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- *An operator* γ *is* increasing *if*
  - $\forall X, Y \in \mathcal{P}(E),\ X \subseteq Y \implies \gamma(X) \subseteq \gamma(Y)$
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Property

- γ *is increasing* $\Leftrightarrow \star\gamma$ *is increasing*
- γ *is idempotent* $\Leftrightarrow \star\gamma$ *is idempotent*

Example. Are the operators of previous examples extensive, increasing and idempotent?

Algebraic dilation and erosion

Definition

- *Let* δ *and* ϵ *be two operators*
- δ *is an* (algebraic) dilation *whenever it commutes under union:*
  - $\forall X, Y \in \mathcal{P}(E)$, $\delta(X) \cup \delta(Y) = \delta(X \cup Y)$
- ϵ *is an* algebraic erosion *whenever it commutes under intersection:*
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- δ *is a dilation if and only if* $\star\delta$ *is an erosion*
- δ *and* ϵ *are increasing*

Example

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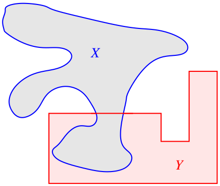
Exercice.

1. Verify the dilation property of γ^r and the erosion property of $\star\gamma^r$ on the following example



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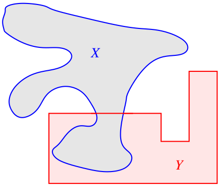

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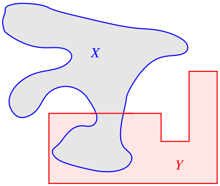

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- Thus, the operator $\star ch$ is not an erosion

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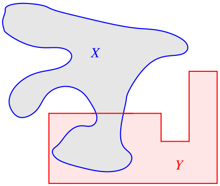

Discrete morphology

Problem

- *How can we define an operator that can handle geometric data (such as images for instance) stored in a computer memory?*
- *How can we efficiently implement such operators?*

Morphological dilation

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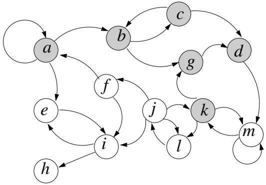
Remark. Any morphological dilation is an algebraic dilation

Straightforward property.
δ_Γ is extensive if and only if (E, Γ) is reflexive
Thus, $\star\delta_\Gamma$ is anti-extensive if and only if Γ is reflexive

Example in an arbitrary graph



$X = \{a, b, c, d, g, k\}$

- $\delta_\Gamma(X) = ?$

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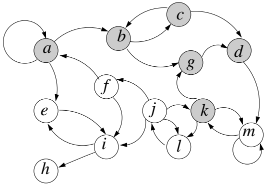

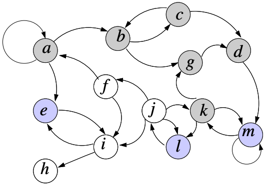

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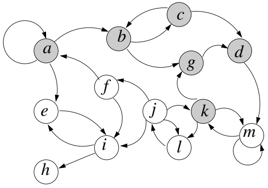

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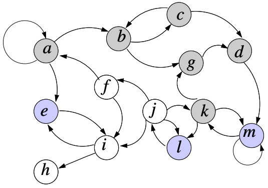

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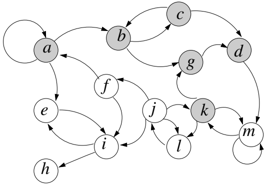

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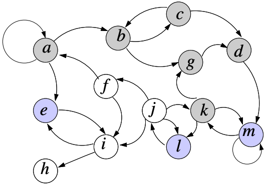

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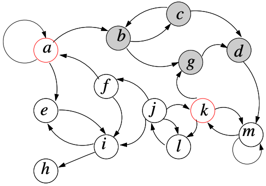

$\star\delta_\Gamma(X) = X \setminus \{k, a\}$

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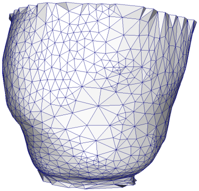
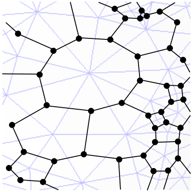
Mesh: example





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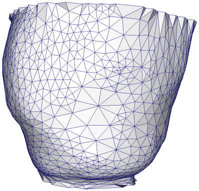

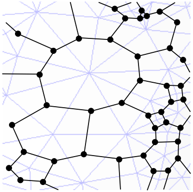

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  - $\forall x, y \in E$, $\Gamma(y) = \mathcal{T}_{\overrightarrow{xy}}(\Gamma(x))$

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Remark.

1. In order to define a translation invariant structuring element, defining $\Gamma(x)$ at a unique point $x \in E$ is sufficient
2. If Γ is translation invariant, then $\forall X \in \mathcal{P}(E)$, $\forall \overrightarrow{v} \in E \times E$, $\delta_\Gamma(\mathcal{T}_{\overrightarrow{v}}(X)) = \mathcal{T}_{\overrightarrow{v}}(\delta_\Gamma(X))$

Example: square grid

- Let $E = \mathbb{Z}^2$ and Γ be defined by $\forall x = (i,j) \in \mathbb{Z}^2$,
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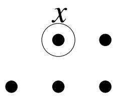
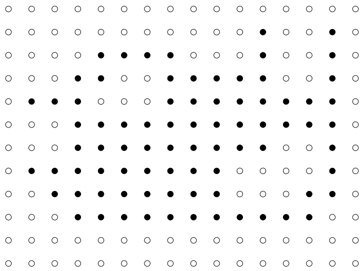


Γ



X (black)

Questions.

1. Use the representation above to draw the structuring elements Γ^{-1} and Γ_s (symmetric closure of Γ).
2. Draw $\delta_\Gamma(X)$, $\delta_{\Gamma^{-1}}(X)$, $\delta_{\Gamma_s}(X)$, $\star\delta_\Gamma(X)$, $\star\delta_{\Gamma^{-1}}(X)$, and $\star\delta_{\Gamma_s}(X)$.

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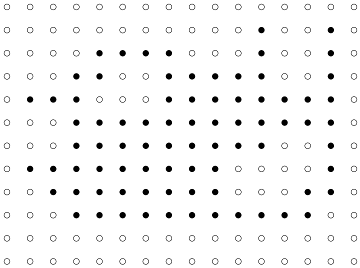

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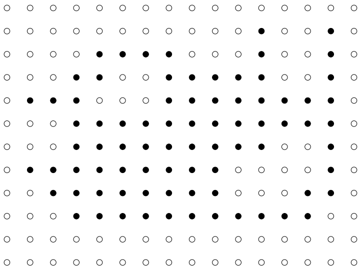

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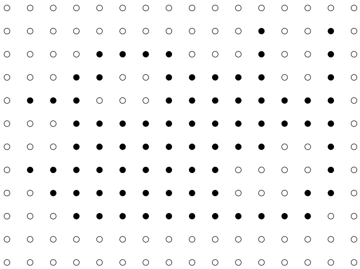

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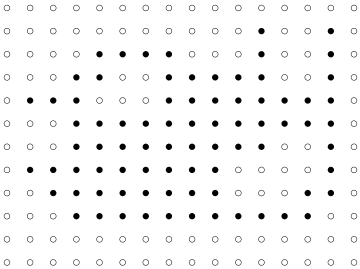
2. Draw $\delta_\Gamma(X)$, $\delta_{\Gamma^{-1}}(X)$, $\delta_{\Gamma_s}(X)$, $\star\delta_\Gamma(X)$, $\star\delta_{\Gamma^{-1}}(X)$, and $\star\delta_{\Gamma_s}(X)$.

Example: square grid

- Let $E = \mathbb{Z}^2$ and Γ be defined by $\forall x = (i,j) \in \mathbb{Z}^2$, $\Gamma(x) = \{(i,j), (i+1,j), (i+1,j-1), (i,j-1), (i-1,j-1)\}$



Γ $\qquad$ X (black)

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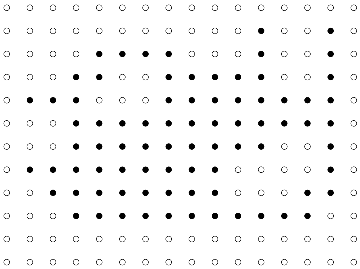

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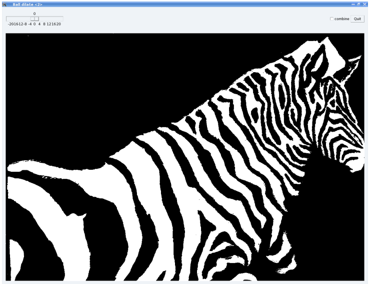
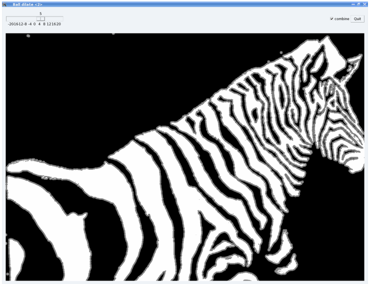
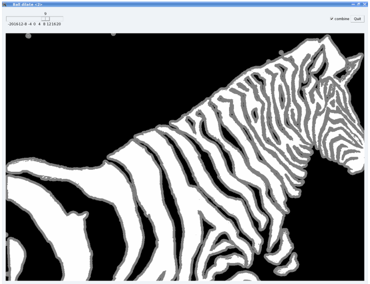
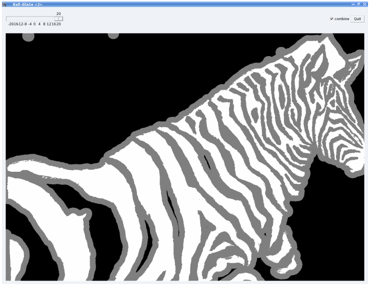
2D imagery: example

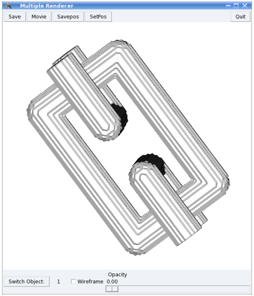


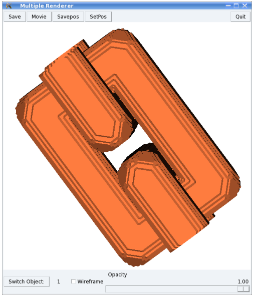


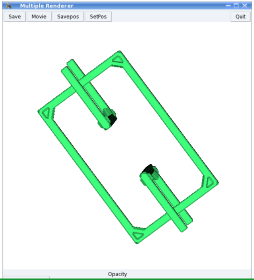


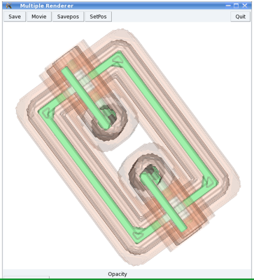


3D imagery: example









Morphological dilation and erosion: characterizations

Property

- *Let* Γ *be a structuring element and let* $X \subseteq E$
  1. $\delta_\Gamma(X) = \{x \in E \mid \Gamma^{-1}(x) \cap X \neq \emptyset\}$

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Proof.

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   $= \{y \in \mid \exists x \in X, x \in \Gamma^{-1}(y)\} = \{y \in E \mid \Gamma^{-1}(y) \cap X \neq \emptyset\}$

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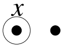
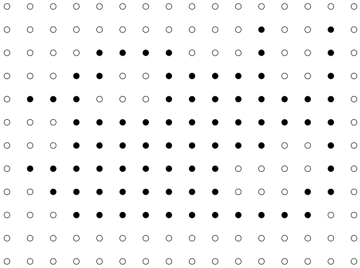
2. $\star\delta_\Gamma(X) = \overline{\delta(\overline{X})}$. Thus, by 1, we deduce :
   $\star\delta_\Gamma(X) = \overline{\{x \in E \mid \Gamma^{-1}(x) \cap \overline{X} \neq \emptyset\}} = \{x \in E \mid \Gamma^{-1}(x) \cap \overline{X} = \emptyset\}$
   $\star\delta_\Gamma(X) = \{x \in E \mid \Gamma^{-1}(x) \subseteq X\}$

Influence of the structuring element

Question. Erode the set X represented in black by the structuring element Γ, that is to say draw $\star\delta_\Gamma(X)$



Γ



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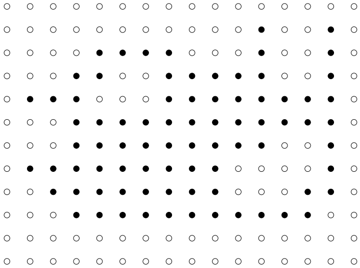

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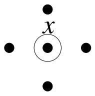

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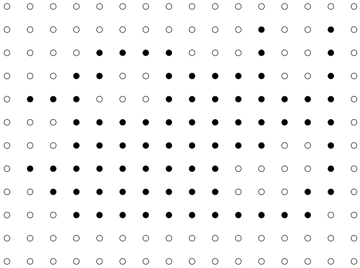

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Structuring elements

- The result of a dilatation/erosion highly depends on the structuring elements that can be
  - of various **shapes**
  - of various **sizes**
  - **isotropic** or not
  - **symmetric** or not
  - **reflexive** or not
  - **convex** or not
  - **translation invariant** or not

Computing a morphological dilation

Algorithm DIL (Data: (E, Γ), $X \subseteq E$; **Result**: $Y = \delta_\Gamma(X)$**)**

- $Y := \emptyset$;
- **For each** $x \in X$ **do**
  - **For each** $y \in \Gamma(x)$ **do** $Y := Y \cup \{y\}$;

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Complexity

- *Algorithm DIL can be implemented to run in* $O(n+m)$ *time, where* $n = |E|$ *and* $m = |\overline{\Gamma}|$
- *Which data structure for* (E,Γ), X *and* Y*?*

Union of structuring elements

Property

- *Let* Γ_1, Γ_2 *and* Γ_3 *be three structuring elements such that* $\overrightarrow{\Gamma_3} = \overrightarrow{\Gamma_1} \cup \overrightarrow{\Gamma_2}$
- $\forall X \in \mathcal{P}(E)$,
  1. $\delta_{\Gamma_3}(X) = \delta_{\Gamma_1}(X) \cup \delta_{\Gamma_2}(X)$
  2. $\star\delta_{\Gamma_3}(X) = \star\delta_{\Gamma_1}(X) \cap \star\delta_{\Gamma_2}(X)$

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Proof.

1. $\delta_{\Gamma_3}(X) = \cup\{\Gamma_3(x) \mid x \in X\}$ Thus, by union of graphs, $\delta_{\Gamma_3}(X) = \cup\{\Gamma_1(x) \cup \Gamma_2(x) \mid x \in X\}$ $= [\cup\{\Gamma_1(x) \mid x \in X\}] \cup [\cup\{\Gamma_2(x) \mid x \in X\}] = \delta_{\Gamma_1}(X) \cup \delta_{\Gamma_2}(X)$
2. By duality, $\star\delta_{\Gamma_3}(X) = \overline{\delta_{\Gamma_3}(\overline{X})}$. Thus, from relation 1., $\star\delta_{\Gamma_3}(X) = \overline{\delta_{\Gamma_1}(\overline{X}) \cup \delta_{\Gamma_2}(\overline{X})}\} = \overline{\delta_{\Gamma_1}(\overline{X})} \cap \overline{\delta_{\Gamma_2}(\overline{X})}\}$. Hence, by duality, $\star\delta_{\Gamma_3}(X) = \star\delta_{\Gamma_1}(X) \cap \star\delta_{\Gamma_2}(X)$ □

Dilations of structuring elements

Definition

- *Let* Γ_1 *and* Γ_2 *be two structuring elements*
- *We denote by* $\Gamma_1 \oplus \Gamma_2$ *the structuring element such that*
  - $\forall x \in E,\ \Gamma_1 \oplus \Gamma_2(x) = \delta_{\Gamma_2}(\Gamma_1(x))$

Composition of dilations by structuring elements

Property

- *Let Γ_1 and Γ_2 be two structuring elements*
  1. $\forall X \in \mathcal{P}(E)$, $\delta_{\Gamma_1 \oplus \Gamma_2}(X) = \delta_{\Gamma_2}(\delta_{\Gamma_1}(X))$
  2. $\forall X \in \mathcal{P}(E)$, $\star\delta_{\Gamma_1 \oplus \Gamma_2}(X) = \star\delta_{\Gamma_2}(\star\delta_{\Gamma_1}(X))$

Proof.

1. $\delta_{\Gamma_1 \oplus \Gamma_2}(X) = \cup\{\Gamma_1 \oplus \Gamma_2(x) \mid x \in X\} = \cup\{\delta_{\Gamma_2}(\Gamma_1(x)) \mid x \in X\}$.
   Since δ_{Γ_2} is an algebraic dilation, δ_{Γ_2} commutes under union.
   Hence, $\delta_{\Gamma_1 \oplus \Gamma_2}(X) = \delta_{\Gamma_2}(\cup\{\Gamma_1(x) \mid x \in X\}) = \delta_{\Gamma_2}(\delta_{\Gamma_1})(X))$
2. The second relation follows from the first one by duality. $\square$

Decomposition of structuring elements

- The previous property shows that a dilation by a “large” structuring element can sometimes be replaced by a composition of dilations by simpler structuring elements
- This can lead to a significant efficiency improvement of some dilation algorithms

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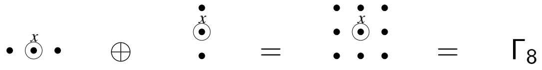

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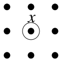
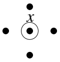
 $\oplus$  $=$  $= \quad \Gamma_8$

But $\Gamma_4 \quad =$  **cannot be decomposed**

Iterated operators

Definition

- *Let* γ *be an operator and let* $i \in \mathbb{N}$
- γ^i *is the operator defined by*
  1. $\gamma^i = \gamma\gamma^{i-1}$
  2. $\gamma^0 = Id$ *(i.e.* $\forall X \subseteq E, \gamma^0(X) = X$*)*

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Property

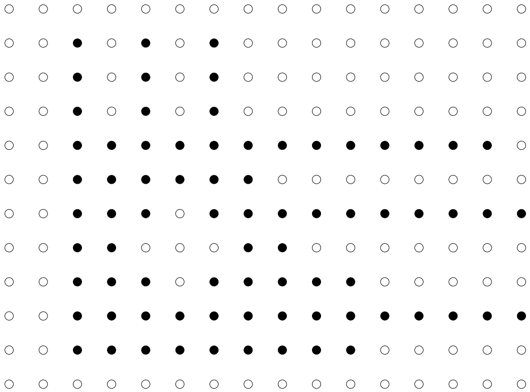
- $\star[\gamma^i] = [\star\gamma]^i$
- *Let* Γ *be a structuring element,* $[\delta_\Gamma]^i = \delta_{\Gamma\oplus\ldots\oplus\Gamma}$

Exercise

- Let $E = \mathbb{Z}^2$ and let Γ_4 be defined by $\forall x = (i,j) \in E, \Gamma_4(x) = \{(i,j),(i-1,j),(i,j-1),(i+1,j),(i,j+1)\}$
- How many elements belong to $\Gamma_4 \oplus \Gamma_4 \oplus \Gamma_4(x)$, for any $x \in E$?
- Compare approximately the number of operations required to compute $\delta_{\Gamma_4 \oplus \Gamma_4 \oplus \Gamma_4}$ and $[\delta_{\Gamma_4}]^3$ by using algorithm DIL
- Indication: you can consider that DIL uses $n + m$ operations to perform a dilation by a structuring element Γ_4 (with $n = |E|$ and $m = |\overrightarrow{\Gamma_4}|$)

Exercise

- Let $X \subseteq \mathbb{Z}^2$ be the black object drawn below
- Which operator (or composition of operators) can you use to suppress the horizontal wire while preserving the vertical ones?
- Which operator (or composition of operators) allows the "hole" of X to be filled in while minimizing the difference between the result and X?



Exercise

- Let $E = \mathbb{Z}^2$, let $X \subseteq E$, and let $\overrightarrow{xy} \in E \times E$ be any vector of $\mathbb{Z}^2$, with $x = (i_x, j_x)$ and $y = (i_y, j_y)$
- The morphological machine can only perform the following operations
    - dilation by a structuring element
    - complementation
- Is it possible to compute the translation of X by $\overrightarrow{xy}$ with the morphological machine?
- Same question for a restricted morphological machine for which the structuring elements must be included in Γ_4

Adjunction

Problem

- *Is there an inverse operator δ' for any dilation δ?*
- *In other words, can we find δ' such that $\forall X \in \mathcal{P}(E), \delta'(\delta(X)) = X$?*

Solution

- *Come to next lecture about mathematical morphology!*

List of main notions

- Increasing, extensive, anti-extensive idempotent operators
- Dual operators
- Algebraic dilation/erosion
- Morphological dilation/erosion (by a structuring element)