Shape Spaces Les Espaces de Formes

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Motivation



Input image.

Result.

Question

How to obtain such a result?



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1 Shape-spaces and connected filtering

2 Shape-based morphology

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1 Shape-spaces and connected filtering

2 Shape-based morphology

3 Some illustrations and applications

1 Shape-spaces and connected filtering

2 Shape-based morphology

3 Some illustrations and applications



1 Shape-spaces and connected filtering

2 Shape-based morphology

3 Some illustrations and applications



Decomposition into primitive or fundamental elements that can be more easily interpreted:

- Functional decompositions;
- Multiresolution decompositions;
- Multi-scale representations;
- Threshold decompositions;
- Hierarchical representations.



Amplitude

Phase

- Functional decompositions;
- Multiresolution decompositions;
- Multi-scale representations;
- Threshold decompositions;
- Hierarchical representations.



- Functional decompositions;
- Multiresolution decompositions;
- Multi-scale representations;
- Threshold decompositions;
- Hierarchical representations.



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- Multiresolution decompositions;
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- Threshold decompositions;
- Hierarchical representations.



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Decomposition into primitive or fundamental elements that can be more easily interpreted:

- Functional decompositions;
- Multiresolution decompositions;
- Multi-scale representations;
- Threshold decompositions;
- Hierarchical representations.

Not mutually exclusive.

Properties inherited from those of underlying operations.

Choice driven by the application needs.

Connected operators

What's connected operators ?

Filtering tools that merge flat zones.

Properties

- No new contours,
- Keep contours' position.

An example : Levelings

Lower-leveling: for x and y neighbors, $g(x) > g(y) \Rightarrow g(y) \ge f(y)$. Upper-leveling: for x and y neighbors, $g(x) > g(y) \Rightarrow g(x) \le f(x)$. Leveling: Lower-leveling \cap Upper-leveling.



Leveling with marker. f : input, h : marker, g : result.

One popular implementation [Salembier & Wilkinson, SPM, 2009]



Level sets and components



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Level sets and components



 $F(x) \geq 3.$

Level sets and components



Level sets and components





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 $F_5 = \{x \mid F(x) \ge 5\}.$



 $F_4 = \{ x \mid F(x) \ge 4 \}.$



 $F_3 = \{x \mid F(x) \ge 3\}.$



 $F_2 = \{x \mid F(x) \ge 2\}.$



 $F_1 = \{x \mid F(x) \ge 1\}.$



 $F_0 = \{x \mid F(x) \ge 0\}.$



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Some of the many possible trees



Some link with Morse's Theory

Important idea

Some nodes are more important than others

- Leaves ⇔ "Extrema"
- Nodes with more than one child ⇔ "Saddle points"
- Hence, filtering is linked with topological persistence

Shape spaces: what are they?

A family of shapes (first part of the talk)

- One can process each shape individually (keep/remove/highlight)
- with any criterion, attribute, energy $\ldots => NOT ROBUST$
- With a tree structure (first part of the talk)
 - A first "topology" on the family of shapes
 - Increasing criterion
- With a graph structure (second part of the talk)
 - A more complete structure on the family of shapes
 - Generalization of the previous approaches
 - With any criterion, attribute, energy ... => **ROBUST**



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A connected component.

Shape-spaces and connected filtering

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Area.

Shape-spaces and connected filtering

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Height.

Shape-spaces and connected filtering

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Attributes

Increasing attributes

Increasing attributes :
$$A \subseteq B \Rightarrow \mathcal{A}(A) \leq \mathcal{A}(B)$$
.
Examples : Area, height, volume.

Non-increasing attributes

Shape attributes.

- I/A^2 minimum for a round object,
- Circularity : $area/(\pi \times l_{max}^2)$,
- Elongation : L_{max}/L_{min} .

 L_{min} and L_{max} : Length of the two main axes of the best fitting ellipse.



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The 9+2 microtubule doublets of a motile cilia Attribute: Volume



Ship detection on optical satellite image

C. Corbane et al., International Journal of Remote Sensing 31 (22), 5837-5854

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SPAC Attack!

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Box detection on a document image Attribute: (width, height) of the component



Line detection on a document image Attribute: (width, height) of the component Operator: top-hat

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Letter detection on a document image Attribute: area of the component Operator: top-hat

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Some functions and spaces on which to compute the trees

- An image
- A gradient
 - Especially usefull with the watershed
- A node-weighted graph
- An edge-weighted graph
- A weighted mesh
- A density function
 - Topological mean-shift filtering
- and more in the sequel















Important idea

- There exists numerous criterions for flooding a surface.
- Flooding can be done through the min-(component-)tree.
- Among those criterions, notably: depth, surface, volume.
- [Beucher, ISMM, 1994 Najman & Schmitt, PAMI, 1996 Meyer et al., An. Telecom, 1997]

Flooding and the min-tree









S. Philipp-Foliguet et al., Pat. Rec., 2011, 44 (3), pp. 588-597



A topological mean-shift algorithm [Paris-Durand CVPR 2007]

A Topological Approach to Hierarchical Segmentation Using Mean Shift

> Sylvain Paris Frédo Durand

> > **CVPR 2007**

Pruning the trees

 $\mathcal{A}\uparrow$, Pruning the leaves = Attribute thresholding.

Non-increasing attributes

How to process the filtering?

Filtering with non-increasing attributes [Salembier & Wilkinson, SPM, 2009]

Pruning strategies

- Min,
- Max,
- Viterbi.

Remove the sub-tree rooted in the node.

Attribute thresholding strategies

- Direct,
- Subtractive.

Remove the nodes under the threshold.

Our proposed framework



Our proposed framework [Xu & Géraud & Najman, ICPR, 2012]



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Outline

Shape-spaces and connected filtering

2 Shape-based morphology

3 Some illustrations and applications









Level $\{x|A(x) \leq 4\}$.



Level $\{x|A(x) \leq 5\}$.



Level $\{x|A(x) \leq 6\}$.



Level $\{x|A(x) \leq 7\}$.



Level $\{x|A(x) \leq 8\}$.

Min-tree of a tree-based image representation



Important idea

Computing a Min-Tree on a node-weighted graph instead of a matrix image.

Easy thanks to Higra [Perret & al., SoftwareX, 2019.], the Hierarchical Graph Analysis library https://higra.readthedocs.io/en/stable/.

Morphological shapings











Thresholding.



Our shaping.



L. Najman: Shape Spaces

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Morphological shapings



Input image.





Shaping based on ${\mathcal A}$



Low threshold of \mathcal{A} . Higher threshold of \mathcal{A} .

Shape-based morphology

Morphological shapings



Input image.

Our shaping 2.

Using a combination of attributes \mathcal{A} .

Encompassing classical attribute filtering strategies

Increasing attribute \mathcal{A}

 $\mathcal{TT} = \mathcal{T}.$ No need to check if the attribute is increasing or not.

Attribute thresholding for non-increasing ${\cal A}$

 $\mathcal{AA} = \mathcal{A},$ \mathcal{AA} is the current level of $\mathcal{TT}.$ **Pruning** $\mathcal{TT} =$ **Attribute thresholding.**

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Shape-based lower/upper-levelings

Shape-based lower-levelings

 \mathcal{T} : Max-tree, $\forall x \in E, \psi_s(f)(x) \le f(x)$ always holds $\Rightarrow \psi_s(f)$ is a lower-leveling of f. \Rightarrow Shape-based lower-levelings.

Shape-based upper-levelings

 \mathcal{T} : Min-tree, $\forall x \in E, \psi_s(f)(x) \ge f(x)$ always holds $\Rightarrow \psi_s(f)$ is a upper-leveling of f. \Rightarrow Shape-based upper-levelings.

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Morphological shapings

Morphological shapings

 \mathcal{T} : Tree of shapes,

The order between $\psi_s(f)$ and f no more guaranteed, not levelings, but it is self-dual.

⇒ Self-dual morphological shapings.



Given a strict order for the set of minima : $A \prec C \prec B$.



B merges with C.



C merges with A.

Strategy

Preserve the **blobs of minima** whose extinction value > a given value.

Advantage

Only the connected components being meaningful enough compared with their context are preserved.



Extinction value of three minima.

Application to object segmentation

Context-based estimator for object detection

[Xu & Géraud & Najman, ICIP, 2012]

$$E(u, \partial \tau) = E_{int}(u, \partial \tau) + E_{ext}(u, \partial \tau) + E_{con}(u, \partial \tau).$$

$$V(u, \mathcal{R}) = \sum_{p \in \mathcal{R}} (u(p) - \overline{u}(\mathcal{R}))^{2},$$

$$E_{ext}(u, \partial \tau) = \frac{V(u, \mathcal{R}_{in}^{\varepsilon}(\partial \tau)) + V(u, \mathcal{R}_{out}^{\varepsilon}(\partial \tau))}{V(u, \mathcal{R}_{in}^{\varepsilon}(\partial \tau) \cup \mathcal{R}_{out}^{\varepsilon}(\partial \tau))},$$

$$E_{int}(u, \partial \tau) = \sum_{e \in \partial \tau} |curv(u)(e)| / L(\partial \tau),$$

$$E_{con}(u, \partial \tau) = 1 / L(\partial \tau).$$

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Application to object segmentation



Energy in a branch of the tree; blue : our energy; green : snake energy.

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(c)

Object detection principle

Significant minima \Leftrightarrow Objects.

Application to object segmentation

Object detection strategy

Morphological closing in the shape-space: Get rid of the spurious minima.

Any attribute \mathcal{A} can be used.

Object detection results

Context-based energy estimator



Input image.

Objects detected.

Shape-based morphology

Object detection results

Shape attribute



Objects detected using shape attribute. Red ones : circularity-based; Green ones : Inverse elongation-based.



(a) Original image.



(b) Some contours.



(a) Original image.



(b) Some contours.



(a) Original image.



(b) Some contours.

Stacking the contours gives a saliency map [Najman & Schmitt, PAMI, 1996]



(a) Original image.



(b) A saliency map.

Different representations

[L. Najman - JMIV - 2011] Mathematical definitions, equivalence between ultrametric watersheds, saliency maps and trees of segmentations



(a) Original image.



(c) One of the segmentations.



(b) Ultrametric watershed.



(d) Dendrogram.

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Saliency maps can be computed on a mesh





Saliency maps from shape-based filterings

Idea

Extinction value for minima \Leftrightarrow Persistence of objects $\stackrel{\mathcal{W}}{\Rightarrow}$ Saliency maps.

Strategy

 $\ensuremath{\mathcal{W}}$: Weight the object contour with the maximum persistence of object that the contour belongs to.

Saliency maps from shape-based filterings





Input image.

Saliency map.

Outline

Shape-spaces and connected filtering

2 Shape-based morphology

3 Some illustrations and applications

4 Conclusion and perspectives

A Topological Approach to Local Feature detection

Interest point detection

- Find a set of interesting points: DoG, Corners, ...
- Find a scale associated to each point

Find interesting regions: MSER[Matas et al., BMVC, 2002]

stability functional τ : $\tau(\mathcal{N}_k) = (|\mathcal{N}_k^+| - |\mathcal{N}_k^-|)/|\mathcal{N}_k|$. |.|: cardinality; \mathcal{N}_k^+ and \mathcal{N}_k^- : resp. ancestor and descendant of node \mathcal{N}_k with a prefixed range of gray level compared with \mathcal{N}_k . Minima of τ are spotted as interesting regions. Generalization: Any tree \mathcal{T} , any attribute \mathcal{A} can be used, and the morphological closing in shape-space filters the meaningless minima.

A Topological Approach to Local Feature detection

Tree-Based Morse Regions (TBMR)



- Select critical nodes (leaves and nodes with several children)
- The scale of a critical node is the largest region containing it and topologically equivalent in its tree.

A Topological Approach to Local Feature detection



Shape-based lower/upper levelings



Input image.



Round objects based upper-leveling.

Shape-based lower/upper levelings



Difference of input image and the shape-based upper-leveling.

Important idea

- 1 Use the green channel,
- 2 Black top-hat transform,
- 3 Extinction-based shape upper-leveling using circularity,
- 4 Preserved connected components are considered as blood vessels.

Tested images

DRIVE database: Digital Retinal Images for Vessel Extraction.

Performances measurements

- 1 Sensitivity and specificity : true positive and negative rate,
- 2 Accuracy: rate of pixels correctly classified,
- 3 kappa value: a statistical measure of inter-rater agreement.

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(d) Shape upper-leveling. (e) Our segmentation. (f) Manual segmentation.



(a) Input image. (b) Input image. (c) Input image. (d) Input image.



(e) Segmentation.(f) Segmentation.(g) Segmentation.(h) Segmentation.

Benchmark on DRIVE database

Method	Sensitivity	Specificity	Accuracy
$2^{\rm nd}$ human observer	0.7761	0.9725	0.9473 (0.0048)
mendonça	0.7344	0.9764	0.9452 (0.0062)
Our	0.6924	0.9779	0.9413 (0.0078)

Benchmark on STARE database

Method	Sensitivity	Specificity	Accuracy
$2^{\rm nd}$ human observer	0.8949	0.9390	0.9354 (0.0171)
Our	0.7149	0.9749	0.9471 (0.0114)
mendonça	0.6996	0.9730	0.9440 (0.0142)

Remark

This is the result of only a "simple" filtering step.

Some illustrations and applications

It also works in 3D: Application to coronary arteries segmentation



Path opening followed by elongation-based filtering

L. Najman: Shape Spaces

Optic nerve head (ONH) segmentation

Important idea

- 1 Use the red channel,
- 2 Classical morphological closing by a 2D disk,
- 3 Construct the tree of shapes and calculate a specific attribute using the fuzzy theory,
- 4 The best filling ellipse of the node having the minimal attribute is identified as the ONH.

Tested images

DRIONS database: Digital Retinal Images for Optic Nerve Segmentation Database.

Performances measurements

Discrepancy.

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Optic nerve head (ONH) segmentation



(a) Input color image.



(b) Red channel.



(c) Results of closing





(d) Detected CC. (e) Segmented ONH. (f) Manual results.

Optic nerve head (ONH) segmentation



(a) Input image. (b) Input image. (c) Input image. (d) Input image.


Optic nerve head (ONH) segmentation

Carmona	96%
Molina	95%
Our	93.6%
Lowell	80%

Percentage of images whose discrepancy is fair

Remark

This is the result of only a "simple" filtering step.

Optic nerve head (ONH) segmentation



Accumulated discrepancy results for our detection method versus Carmona et al, Molina et al and Lowell et al.

Mumford-Shah energy with cartoon model

$$E_{\mathcal{T}} = \sum_{\partial \tau \in \mathcal{T}} \bigg(\sum_{p \in \mathcal{R}(\partial \tau)} \Big(u(p) - \overline{u} \big(\mathcal{R}(\partial \tau) \big) \Big)^2 + \nu L(\partial \tau) \bigg).$$

Attribute

 ν measures the simplification level.

Important idea

- 1 Construct the tree of shapes,
- 2 Weight each node with the simplification level ν ,
- 3 The saliency map yields a hierarchical simplification.

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Original. Saliency map.

Simplified.



Original.

Saliency map.



Original.

Saliency map.



Original.

Saliency map.



Original.

Saliency map.



Original.

Saliency map.

Felzenswalb and Huttenlocher's algorithm

[Felzenswalb & Huttenlocher], IJCV, 2004

- Compute a minimum spanning tree (MST) of a dissimilarity,
- 2 For each edge ∈ MST linking two vertices x and y, in increasing order of their weights:
 - (i) Find the region X that contains x,
 - (ii) Find the region Y that contains y,
 - (iii) Merge X and Y if

$$Diff(X,Y) < \min\{Int(X) + \frac{k}{|X|}, Int(Y) + \frac{k}{|Y|}\}.$$

Question

Is k a scale parameter?

Causality principle

- A contour present at a scale k₁ should be present at any scale k₂ < k₁.
- Not true with Felzenswalb and Huttenlocher's algorithm.



Original.





k = 7500 (8 regions). k = 9000 (14 regions).

Application of our framework with attribute k

Answer

k is not a scale parameter.

Attribute from k

$$k = \max \left\{ \left(\textit{Diff}(X, Y) - \textit{Int}(X) \right) imes |X|, \left(\textit{Diff}(X, Y) - \textit{Int}(Y) \right) imes |Y|
ight\}.$$

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Hierarchical image segmentation on BSDS500

Important idea

- 1 Calculate the distance between neighboring pixels,
- 2 Construct a minimum spanning tree (MST),
- 3 Compute attribute k,
- 4 The saliency map yields an hierarchical image segmentation.

Tested images

BSDS500: Berkeley Segmentation Data Set and Benchmarks 500.

Performance measurements

- 1 Ground-truth Covering [Arbeláez et al., PAMI, 2011],
- 2 Probabilistic Rand Index [Arbeláez et al., PAMI, 2011].

Hierarchical image segmentation on BSDS500



Some illustrations and applications

Hierarchical image segmentation on BSDS500







Original.

Saliency map.

Segmentation(70 regions).

Hierarchical image segmentation on BSDS500







Original.

Saliency map.

Segmentation(20 regions).

Hierarchical image segmentation on BSDS500

Benchmarks

Our method obtains better results than the results of method of FH, and of method of Guimarães for optimal dataset scale (ODS), and for optimal image scale (OIS).

Method	GT Covering		Prob. Rand. Index		
	ODS	OIS	Best	ODS	OIS
FH	0.43	0.53	0.68	0.76	0.79
Guimarães	0.46	0.53	0.60	0.76	0.81
Ours	0.50	0.57	0.66	0.77	0.82

Comparison of the hierarchical segmentation obtained with Felzenswalb and Huttenlocher's algorithm, method of Guimarães et al., and our method.

Outline

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Conclusion

Object filtering

- 1 Encompass the state of art,
- 2 Shape-based lower/upper-levelings,
- 3 Morphological shapings.
- Object detection
- 1 Context-based estimator,
- $2\,$ Specific attribute ${\cal A}$ for ONH segmentation,
- 3 Saliency map.

- Attributes \mathcal{A} and $\mathcal{A}\mathcal{A}$,
- Learning of the attributes,
- Strategies of dealing with second tree \mathcal{TT} ,
- More Properties of the morphological shapings,
- Saliency maps.

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



The model

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Labeling with super-pixels

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Labeling with a CRF regularization

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Labeling with an optimal cover tree

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Optimal cover: $\{C_1, C_3, C_4, C_5\}$

The optimal cover tree

Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Learning Hierarchical Features for Scene Labeling

C. Farabet et al., PAMI 2013



Thank for your attention !





Mathematical Morphology

Edited by Laurent Najman and Hugues Talbot



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Pink:http://pinkhq.comHigra:https://higra.readthedocs.io/