Introduction to morphological filtering

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MorphoGraph

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Outline of the lecture

1. Filters: openings and closings
2. Openings and closings by adjunction
3. Openings and closings by structuring elements

Filter

Definition

- *A* filter (on E) *is an operator* γ *that is both increasing and idempotent*
  - $\forall X, Y \in \mathcal{P}(E),\ X \subseteq Y \implies \gamma(X) \subseteq \gamma(Y)$
  - $\forall X \in \mathcal{P}(E),\ \gamma(\gamma(X)) = \gamma(X)$

Closing and opening

Definition

- *A* closing (on E) *is a filter* γ *that is extensive*
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- *An* opening (on E) *is a filter* γ *that is anti-extensive*
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Property

- γ *is a closing if and only if* $\star\gamma$ *is an opening*

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Exercise.

- Prove this property by establishing the three following relations
  - $\forall X, Y \in \mathcal{P}(\mathbb{R}^2),\ X \subseteq Y \implies ec(X) \subseteq ec(Y)$ (increasingness)
  - $\forall X \in \mathcal{P}(\mathbb{R}^2),\ ec(X) = ec(ec(X))$ (idempotence)
  - $\forall X \in \mathcal{P}(\mathbb{R}^2),\ X \subseteq ec(X)$ (extensivity)

Example 1

Property

- *The convex hull operator ch on $\mathbb{R}^2$ is a closing*
- *The operator $\star ec$ is then an opening*

Exercise.

- Prove this property by establishing the three following relations
  - $\forall X, Y \in \mathcal{P}(\mathbb{R}^2)$, $X \subseteq Y \implies ec(X) \subseteq ec(Y)$ (increasingness)
  - $\forall X \in \mathcal{P}(\mathbb{R}^2)$, $ec(X) = ec(ec(X))$ (idempotence)
  - $\forall X \in \mathcal{P}(\mathbb{R}^2)$, $X \subseteq ec(X)$ (extensivity)

Adjunction issue

- The notion of an adjunction plays a central role in morphology
- It allows an opening and a closing to be built from any dilation (*i.e.*, an operator that commutes under union)

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Question

- *Given a dilatation δ, can we always find an inverse operator δ' to δ?*
- *In other words, can we find δ' such that*
  - $\forall X \in \mathcal{P}(E), \delta(\delta'(X)) = X$?

δ-lower set

Definition

- *Let δ be a dilation and let $X, X' \in \mathcal{P}(E)$*
- *X' is* δ-lower than X *if*
    - $\delta(X') \subseteq X$

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Property

- *Let δ be a dilation and let $X \in \mathcal{P}(E)$*
- *Among the sets that are δ-lower than X, there exists a greatest element $\dot{X}$*
    - $\dot{X} = \cup\{X' \in \mathcal{P}(E) \mid X' \text{ is } \delta\text{-lower than } X\}$

Proof

By definition of union, $\dot{X}$ is the smallest set that contains all the sets that are δ-lower than X. In order to complete the proof, it is sufficient to show that $\dot{X}$ is also δ-lower than X (*i.e.* $\delta(\dot{X}) \subseteq X$). By definition of a set that is δ-lower than X, we can write $\dot{X} = \cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq X\}$.
Thus, $\delta(\dot{X}) = \delta(\cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq X\})$. Since the dilation operator commutes under union, we can also write $\delta(\dot{X}) = \cup\{\delta(X') \mid X' \in \mathcal{P}(E), \delta(X') \subseteq X\}$. Therefore, by definition of union, we have the relation $\delta(\dot{X}) \subseteq X$, which completes the proof of the property. □

Proof

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Exercise. Prove that, in general, there is no smallest element among the sets that are "δ-greater" than X.

Adjunct erosion

Definition

- *Let δ be a dilation*
- *The* adjunct erosion of δ *is the operator $\dot{\delta}$ that maps any $X \in \mathcal{P}(E)$ to the greatest element in $\mathcal{P}(E)$ that is δ-lower than X :*
  - $\dot{\delta}(X) = \cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq X\}$

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Theorem

- *If δ is a dilation, then $\dot{\delta}$ is an erosion (i.e., $\dot{\delta}$ commutes under intersection)*

Proof of the theorem

Proof. Let $A, B \in \mathcal{P}(E)$. $\dot{\delta}(A \cap B) = \cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq [A \cap B]\}$
$= \cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq A \text{ et } \delta(X') \subseteq B\}$
$= [\cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq A\}] \cap [\cup\{X' \in \mathcal{P}(E) \mid \delta(X') \subseteq B\}]$
$= \dot{\delta}(A) \cap \dot{\delta}(B)$ □

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- If ϵ is an erosion, its *adjunct dilation* $\dot{\epsilon}$ is defined by:
  - $\forall X \in \mathcal{P}(E)$, $\dot{\epsilon}(X) = \cap\{X' \mid X \subseteq \epsilon(X')\}$

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  - $\epsilon = \dot{\delta} \Leftrightarrow \delta = \dot{\epsilon}$
- $\dot{\delta} \circ \delta = Id \Leftrightarrow \delta = \dot{\delta} = Id$

Adjunctions by structuring elements

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Important notation

- If Γ is a structuring element
- We denote by ϵ_Γ the adjunct erosion of δ_Γ
  - $\epsilon_\Gamma = \dot{\delta_\Gamma} = \star\delta_{\Gamma^{-1}}$

Closing and opening by adjunctions

Theorem

- *Let* δ *be a dilation and let* $\epsilon = \dot{\delta}$ *be the adjunct erosion of* δ
- *Let* $\phi = \epsilon \circ \delta$ *and* $\gamma = \delta \circ \epsilon$

Closing and opening by adjunctions

Theorem

- *Let* δ *be a dilation and let* $\epsilon = \dot{\delta}$ *be the adjunct erosion of* δ
- *Let* $\phi = \epsilon \circ \delta$ *and* $\gamma = \delta \circ \epsilon$
- ϕ *is a closing*
- γ *is an opening*

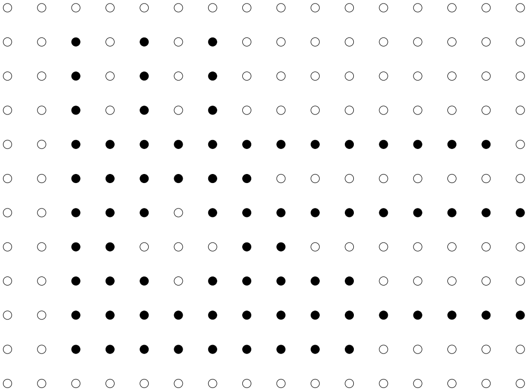
Opening and closing by structuring elements

Definition

- *Let* Γ *be a structuring element*
- *The* closing by Γ *is the operator* ϕ_Γ *such that*
  - $\phi_\Gamma = \star\delta_{\Gamma^{-1}} \circ \delta_\Gamma$
- *The* opening by Γ *is the operator* γ_Γ *such that*
  - $\gamma_\Gamma = \delta_\Gamma \circ \star\delta_{\Gamma^{-1}}$

Exercise 1

- Let $X \subseteq \mathbb{Z}^2$ be the set of black dots and let Γ be the structuring element below
- Represent the set $\gamma_\Gamma(X)$



Γ :

Characterization of the opening/closing by structuring element

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Topographical interpretation

- We say that $X \in \mathcal{P}(E)$ is *thinner* than the structuring element Γ if $\star\delta_{\Gamma}(X) = \emptyset$

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  - islands (isolated parts)
  - capes (thin convexities)
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- The opening of the set X by Γ removes the parts of X that are thinner than Γ, that is to say
  - islands (isolated parts)
  - capes (thin convexities)
  - isthmus (junctions between non thin parts)
- The closing removes the thin parts of $\overline{X}$, that is to say
  - lakes (holes)
  - gulfs (thin concavities)
  - straits (junctions between non thin parts of $\overline{X}$)

Exercise 2

- Choose and apply an operator that “fills” the holes of the black object X below